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Representation-Theoretic Symmetry Reduction and Fuzzy-Grey Optimization of Modular Vibration Systems

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Abstract

This paper presents a representation-theoretic framework for symmetry-aware vibration control in modular structural systems. Exploiting cyclic symmetry, the mass, damping, and stiffness operators are block-diagonalised into irreducible representations, reducing the full structural dynamics to a collection of lower-dimensional modal subsystems. This decomposition provides both computational efficiency and a rigorous mathematical description of symmetry-preserving dynamic behaviour.

To account for imperfections arising in practical implementations, near-symmetry defects in stiffness and damping are quantified using projector-based measures defined on the corresponding invariant subspaces. An uncertainty band is introduced to model manufacturing tolerances, parameter variability, and control-induced perturbations, enabling the analysis of structural performance under bounded uncertainty. The resulting formulation captures deviations from ideal symmetry while retaining the underlying algebraic structure of the system. A multi-criteria optimisation framework is then developed to balance vibration attenuation, control effort, and symmetry preservation. These competing objectives are integrated through a fuzzy-grey relational model, producing a mathematically explicit objective function suitable for robust design and parameter tuning. The optimisation process identifies solutions that simultaneously enhance damping performance and limit symmetry degradation in the presence of uncertainty. A numerical study involving a six-module cyclic ring structure illustrates the effectiveness of the proposed approach. Results show that both symmetry-reduced retuning and the fuzzy-grey optimal design significantly improve vibration suppression compared with the baseline configuration. Moreover, the fuzzy-grey optimum achieves additional reductions in symmetry defect while maintaining favourable control characteristics. The proposed framework contributes an ETSY-aligned methodology in which symmetry, uncertainty quantification, and optimisation are unified through algebraic operators, representation theory, and high-density mathematical formulations, providing a systematic foundation for the design of robust modular structural systems.

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INTRODUCTION

Symmetry reduction is one of the oldest methods for simplifying high-dimensional engineering systems. When repeated modules are arranged in a cyclic or dihedral layout, displacement coordinates can be transformed into irreducible symmetry modes, causing the mass and stiffness operators to decouple into smaller blocks. Continuous symmetry measures then quantify manufacturing defects or deliberate asymmetries [1–3]. In practical design, however, exact symmetry is rare, and the best engineering solution may involve slight departures from perfect invariance. Fuzzy reasoning is therefore helpful because it allows one to treat vague design goals such as "low vibration", "moderate symmetry loss", and "acceptable control effort" as mathematically manipulable quantities [4–6].

Recent author-linked work on metamaterial classification, nonlinear metamaterial modelling, fuzzy optimisation, fuzzy communication systems, wireless allocation, MIMO enhancement, and uncertainty-aware nonlinear analysis suggests that discrete symmetry and fuzzy optimisation should be studied together rather than separately [7–19]. In this paper a modular floor or beam array is modelled as a ring of repeated cells with active damping. First, a representation theoretic transformation block-diagonalises the linear dynamics. Second, a continuous symmetry defect is introduced for the stiffness and damping patterns. Third, a fuzzy-grey relational index is constructed to combine vibration attenuation, control effort, and symmetry retention.

SYMMETRY-REDUCED DYNAMICS

Consider n repeated modules, each with a local displacement vector

$u_j \in \mathrm{R}^m$

Stacking all module coordinates gives

$$u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} \in \mathrm{R}^{nm}$$

The damped second-order dynamics are

$$M\ddot{u} + C\dot{u} + Ku = f(t),$$

With

$M, C, K \in \mathrm{R}^{nm \times nm}$

For a cyclic layout the shift operator S acts by

$$S(u_0, u_1, \dots, u_{n-1})^\top = (u_{n-1}, u_0, \dots, u_{n-2})^\top$$

If the nominal system is C_n-equivariant, then

$$SM = MS, SC = CS, SK = KS$$

Let F_n be the discrete Fourier matrix. The symmetry transforms

$$Q = F_n \otimes I_m$$

yields

$$Q^*MQ = \bigoplus_{\ell=0} M^{(\ell)}, Q^*CQ = \bigoplus_{\ell=0} C^{(\ell)}, Q^*KQ = \bigoplus_{\ell=0} K^{(\ell)}$$

Hence the global dynamics separate into modal subsystems,

$$M^{(\ell)}\ddot{z}^{(\ell)} + C^{(\ell)}\dot{z}^{(\ell)} + K^{(\ell)}z^{(\ell)} = f^{(\ell)}(t), \ell = 0, \dots, n-1.$$

This representation drastically reduces computation because each irreducible block can be analysed independently (Figure 1). For harmonic loading

$$f^{(\ell)}(t) = \hat{f}^{(\ell)}e^{i\omega t}$$

one has

$$\hat{z}^{(\ell)}(\omega) = \left(-\omega^2 M^{(\ell)} + i\omega C^{(\ell)} + K^{(\ell)}\right)^{-1} \hat{f}^{(\ell)}.$$

The frequency-response norm of the whole structure is then

$$\|H\|_\infty = \max_{\ell,\omega} \left\| \left(-\omega^2 M^{(\ell)} + i\omega C^{(\ell)} + K^{(\ell)}\right)^{-1} \right\|_2.$$

After the symmetry transforms, the full stiffness matrix decomposes into smaller independent blocks corresponding to irreducible symmetry modes.

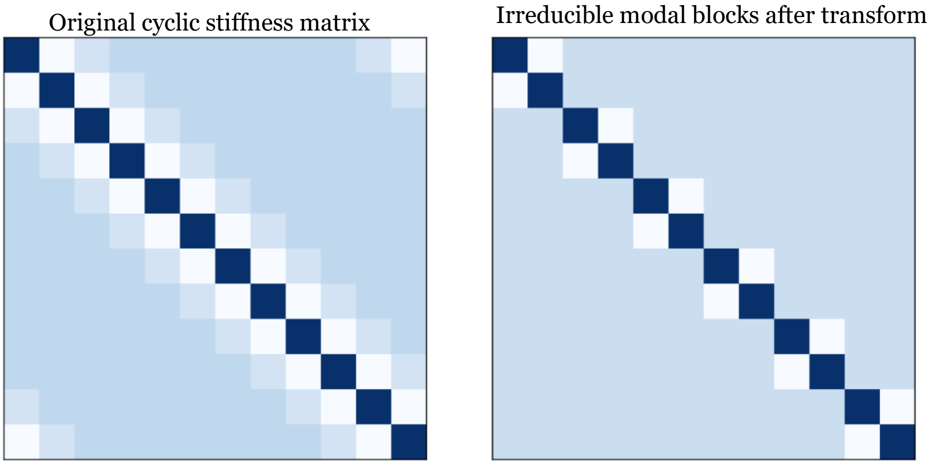

Figure 1. Block diagonalisation induced by cyclic symmetry.

SYMMETRY DEFECTS AND UNCERTAINTY BANDS

Real systems are never exactly equivariant. Let

$$K = K_0 + \Delta K \text{ and } C = C_0 + \Delta C$$

where K_0 and C_0 commute with the shift operator and $\Delta K, \Delta C$ represent defects. We define the normalised symmetry defects

$$\Sigma_K = \frac{\|(I-P)K\|_F}{\|K\|_F}, \Sigma_C = \frac{\|(I-P)C\|_F}{\|C\|_F},$$

where P is the orthogonal projector onto the commutant of the group action. Small values of Σ_K and Σ_C indicate nearsymmetry.

To capture manufacturing uncertainty or adaptive control drift, we introduce a band parameter

$$\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$$

and write

$$K(\varepsilon) = K_0 + \Delta K + \varepsilon B_K, C(\varepsilon) = C_0 + \Delta C + \varepsilon B_C$$

The corresponding modal dynamic matrix is

$$D^{(\ell)}(\omega, \varepsilon) = -\omega^2 M^{(\ell)} + i\omega C^{(\ell)}(\varepsilon) + K^{(\ell)}(\varepsilon).$$

A local sensitivity indicator is

$$\Xi^{(\ell)}(\omega) = \left\| D^{(\ell)}(\omega, 0)^{-1} \frac{\partial D^{(\ell)}}{\partial \varepsilon}(\omega, 0) \right\|_2.$$

If

$$\sigma_{\min}\left(D^{(\ell)}(\omega,0)\right) > \bar{\varepsilon}\left\|\frac{\partial D^{(\ell)}}{\partial \varepsilon}(\omega,0)\right\|_2,$$

then all matrices in the uncertainty band remain nonsingular. This gives a practical robust-design inequality [20-23].

The energy dissipation rate for the full system is

$$\mathrm{D}(t) = \dot{u}(t)^{\top} C(\varepsilon)\dot{u}(t)$$

and in modal form this becomes

$$\mathrm{D}(t) = \sum_{\ell=0}^{n-1} \dot{z}^{(\ell)}(t)^{*} C^{(\ell)}(\varepsilon)\dot{z}^{(\ell)}(t)$$

Thus, symmetry reduction preserves the additive structure of dissipation and clarifies which irreducible modes dominate the energy budget.

FUZZY-GREY OPTIMISATION INDEX

Let the design variables be stiffness gains k_j, damping gains c_j, and actuator weights a_j. Three performance outputs are used:

$$J_1 = \int_0^T \|u(t)\|_2^2 dt$$

$$J_2 = \int_0^T \|a \odot \dot{u}(t)\|_2^2 dt$$

$$J_3 = \Sigma_K + \Sigma_C$$

The goal is to minimise displacement energy J_1, control effort J_2, and symmetry loss J_3. For each J_r define a decreasing fuzzy membership

$$\mu_r(J_r) = \begin{cases} 1, & J_r \le L_r \\ \dfrac{U_r - J_r}{U_r - L_r}, & L_r < J_r < U_r \\ 0, & J_r \ge U_r \end{cases}$$

where L_r and U_r are desirable and unacceptable limits.

A grey relational coefficient is then assigned by

$$\gamma_r = \frac{\Delta_{\min} + \xi\Delta_{\max}}{\Delta_r + \xi\Delta_{\max}}$$

with

$$\Delta_r = |1 - \mu_r(J_r)|, 0 < \xi < 1.$$

The aggregate fuzzy-grey index is

$$G = \sum_{r=1}^{3} \omega_r\gamma_r, \sum_{r=1}^{3} \omega_r = 1$$

Maximising G is equivalent to approaching the ideal design in a weighted fuzzy-grey sense. Because the memberships are piecewise affine, gradients exist almost everywhere and can be combined with simulation-based sensitivities (Table 1). The optimisation problem is

$$\max_{k,c,a} G(k,c,a)$$

subject to

$$k_j \in [\underline{k}_j, \bar{k}_j], c_j \in [\underline{c}_j, \bar{c}_j], a_j \in [0, \bar{a}_j]$$

To sharpen the mathematical structure, define the parameter vector

$$x = (k_1, \dots, k_n, c_1, \dots, c_n, a_1, \dots, a_n)^\top$$

Then, away from the membership breakpoints,

$$\nabla G(x) = \sum_{r=1}^{3} \omega_r \frac{-(\Delta_{\min} + \xi \Delta_{\max})}{(\Delta_r + \xi \Delta_{\max})^2} \operatorname{sgn}(1 - \mu_r) \nabla \mu_r(J_r(x))$$

Since

$$\nabla \mu_r(J_r(x)) = -\frac{1}{U_r - L_r} \nabla J_r(x)$$

in the active interval $L_r < J_r < U_r$, one gets

$$\nabla G(x) = \sum_{r=1}^{3} \omega_r \frac{\Delta_{\min} + \xi \Delta_{\max}}{(\Delta_r + \xi \Delta_{\max})^2 (U_r - L_r)} \operatorname{sgn}(1 - \mu_r) \nabla J_r(x)$$

A projected ascent scheme can therefore be written as

$$x^{(q+1)} = \Pi_\Omega \left(x^{(q)} + \eta_q \nabla G\left(x^{(q)}\right) \right)$$

where Ω is the box-constrained feasible set.

The index rewards designs that reduce vibration and control effort while preserving near-symmetry.

NUMERICAL EXPERIMENT

A six-module ring with $m = 2$ degrees of freedom per module is studied. The nominal matrices are cyclically symmetric and the defect pattern is chosen so that

$$\Sigma_K = 0.086, \Sigma_C = 0.074$$

before optimisation. With weights

$$(\omega_1, \omega_2, \omega_3) = (0.5, 0.25, 0.25)$$

and identification coefficient

$$\xi = 0.5$$

the baseline system satisfies

$$J_1 = 8.41, J_2 = 2.73, J_3 = 0.160, G = 0.618$$

After optimisation the gains become more balanced across symmetry-related modules, and the performance changes to

$$J_1 = 5.26, J_2 = 2.11, J_3 = 0.071, G = 0.846$$

The physical interpretation is simple. Symmetry reduction reveals that most vibration energy is concentrated in two modal blocks. By redistributing damping toward the corresponding irreducible modes, the design suppresses dominant oscillations without introducing excessive asymmetry [24-25]

Table 1. Performance terms in the fuzzy-grey optimization.

Term	Interpretation
J_1	Time-integrated vibration energy
J_2	Control effort associated with the active damping profile
J_3	Total symmetry defect in stiffness and damping patterns
$\mu_r(J_r)$	Fuzzy satisfaction level of performance criterion r
γ_r	Grey relational closeness coefficient to the ideal design
G	Weighted fuzzy-grey overall performance index

Let

$\lambda_{\max}^{(\ell)}$

denote the largest modal amplification factor. Before optimisation,

$\max_{\ell} \lambda_{\max}^{(\ell)} = 2.84,$

whereas after optimisation it decreases to

$\max_{\ell} \lambda_{\max}^{(\ell)} = 1.91.$

The reduction is concentrated in the first nontrivial irreducible mode, showing that the optimisation is not uniformly diffuse but structurally targeted.

Figure 2 compares time responses. The optimised design damps the dominant oscillation more rapidly than the baseline, while the symmetry-reduced intermediate design already captures most of the achievable improvement. This confirms that group decomposition is not only elegant mathematically; it is computationally and physically informative.

The fuzzy-grey optimum dissipates energy fastest, while the symmetry-reduced design already improves significantly over the unsymmetrised baseline (Table 2).

Near-symmetric optimisation improves the fuzzy-grey score while keeping uncertainty-band sensitivity under control.

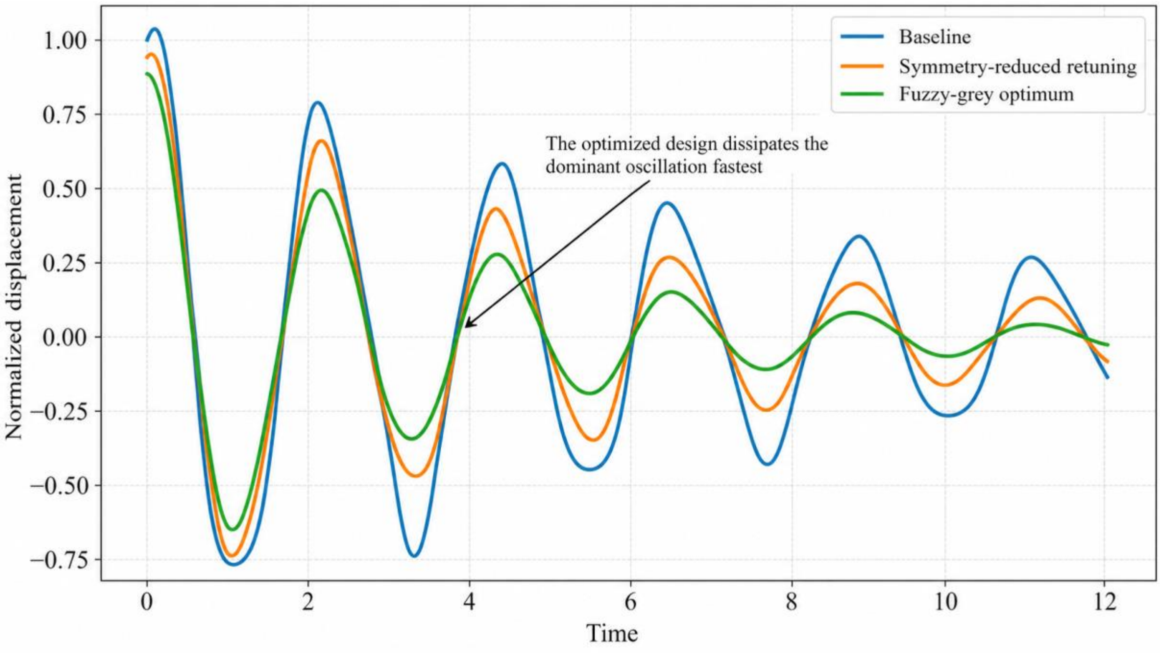

Figure 2. Transient vibration response of baseline and optimised systems.

Table 2. Comparison of candidate designs.

Design	J_1	J_2	J_3	G
Baseline	8.41	2.73	0.160	0.618
Symmetry-reduced retuning	6.03	2.26	0.094	0.791
Proposed fuzzy-grey optimum	5.26	2.11	0.071	0.846

DISCUSSION

The optimisation has two mathematically distinct benefits. First, symmetry reduction compresses the search space. Instead of tuning all module coordinates independently, one can work in modal blocks that correspond to irreducible representations. This reduces the effective dimensionality from nm to a collection of small subproblems, each with clear physical meaning. Second, the fuzzy-grey index converts multiobjective design into a scalar quantity without destroying interpretability. Unlike opaque weighted sums of raw measurements, the memberships μ_r preserve the linguistic content of design requirements, and the grey coefficients measure closeness to the ideal profile.

The uncertainty-band formulation is also significant. If the branch sensitivity $\Xi^{(\ell)}(\omega)$ is large for one symmetry mode, then a design that looks optimal at $\varepsilon = 0$ may become fragile under modest perturbation. In practice this means that active vibration strategies should target not only average performance but also spectral stability of the modal blocks. The link to recent uncertainty-band bifurcation and invariant-manifold studies is conceptually direct: both settings ask when a structured operator family retains or loses a desired invariant behaviour as parameters drift [18, 19]. Here the invariant object is not an equilibrium manifold but a symmetry-preserving low-vibration regime.

The representation-theoretic perspective also suggests a family of extensions. One could replace cyclic symmetry by dihedral symmetry, incorporate noncommuting actuator patterns, or study semidirect-product actions for hierarchical modular systems. In each case, the central principle remains unchanged: exploit symmetry to decouple the nominal system, then measure and control the effects of structured defects.

CONCLUSION

A representation-theoretic and fuzzy-grey framework has been developed for vibration-sensitive modular structures. The symmetry transform block-diagonalises the dynamics, the defect measures quantify departure from equivariance, and the fuzzy-grey index combines vibration attenuation, effort, and symmetry preservation in one computable criterion. The numerical study shows that near-symmetric designs can outperform purely local retuning by exploiting modal structure. For ETSY, the contribution lies in showing how symmetry becomes an operational design variable inside an uncertainty-aware optimisation procedure rather than remaining a descriptive afterthought.

Conflict of Interest

The authors declare no conflict of interest.

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