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Fractional Riemannian Fuzzy C-Means with Time-Series Regularization for Economic Manifold Forecasting

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Abstract

A fractional Riemannian fuzzy c-means framework is proposed for uncertain economic forecasting on non-Euclidean data domains. Observations are represented on a Riemannian manifold, cluster centres are intrinsic prototypes, and a latent fuzzy regime signal is regularised by both autoregressive and fractional-memory penalties. The resulting objective couples geometric clustering with time-series consistency, thereby discouraging partitions that are locally plausible but temporally incoherent. Closed-form membership updates, exponential-map centre updates, normal equations for the predictive coefficients, and a monotone descent property are derived. A manifold-valued forecasting rule is then obtained from predicted fuzzy memberships and intrinsic barycentres. Numerical experiments on stylised quarterly economic compositions show improved regime smoothness and lower forecast error relative to Euclidean fuzzy c-means baselines. The paper contributes a mathematically dense RTM-style model that integrates fuzzy clustering, manifold optimisation, and fractional time-series analysis.

Keywords: Fuzzy c-means, Riemannian manifold, fractional memory, economic forecasting, intrinsic clustering, time series.

INTRODUCTION

Many economic datasets are simultaneously geometric, temporal, and uncertain. The geometric aspect appears when indicators are normalised to probability simplices or covariance-type manifolds, the temporal aspect appears through serial dependence and regime switching, and the uncertainty aspect appears because macroeconomic and behavioural variables are frequently imprecise, incomplete, or linguistically qualified. Fuzzy sets and fuzzy decision environments remain natural tools for such data [1-3]. Fuzzy clustering, beginning with the classical Dunn and Bezdek formulations, provides a graded assignment mechanism superior to hard partitioning when regimes overlap [4,5]. More recent work improved convergence and scalability [6], while manifold-based clustering and optimisation show that Euclidean geometry is not always appropriate for probability-like data [7-9].

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A large recent author-linked literature motivates the present mathematical synthesis. Fuzzy time series were applied to economic trends [10], fuzzy clustering was developed for economic data mining and consumer analysis [11,12], behavioural economics was modelled through fuzzy logic [13], and hybrid crisp-fuzzy, statistical-fuzzy, agricultural, habitat, healthcare, graph-theoretic, and optimisation studies repeatedly emphasised the need for interpretable uncertainty quantification rather than purely black-box learning [14-25].

This paper proposes a fractional Riemannian fuzzy c-means model with time-series regularisation for economic manifold forecasting.

The novelty lies in combining: (i) an intrinsic distance on a manifold, (ii) fractional memory across consecutive time points, and (iii) a fuzzy time-series penalty linking cluster assignments to predictive dynamics.

GEOMETRIC MODEL

Let $x_t \in M$ for $t = 1, \dots, T$ be a sequence on a complete Riemannian manifold (M, g), in the numerical study, M is the multinomial manifold. Let $c_k \in M, k = 1, \dots, C$, denote cluster centres and let $u_{tk} \in [0,1]$ satisfy

$$\sum_{k=1}^{C} u_{tk} = 1, u_{tk} \geq 0$$

The intrinsic squared distance is $d_g(x_t, c_k)^2$. A fractional memory operator is introduced by

$$\Delta^\gamma z_t = \sum_{j=0}^{t-1} \omega_j^{(\gamma)} z_{t-j}, \omega_j^{(\gamma)} = (-1)^j \binom{\gamma}{j}, 0 < \gamma < 1$$

where z_t is a latent regime signal constructed from fuzzy assignments. We set

$$z_t = \sum_{k=1}^{C} k u_{tk}$$

The objective functional is

$$J(U, C, \phi) = \sum_{t=1}^{T} \sum_{k=1}^{C} u_{tk}^m d_g(x_t, c_k)^2 + \lambda \sum_{t=p+1}^{T} \left(z_t - \sum_{j=1}^{p} \phi_j z_{t-j} \right) + \tau \sum_{t=2}^{T} (\Delta^\gamma z_t)^2$$

where $m > 1$ is the fuzzifier, $\phi = (\phi_1, \dots, \phi_p)$ are autoregressive coefficients, and $\lambda, \tau \geq 0$ weight temporal coherence and fractional memory as shown in table 1.

The first term is intrinsic fuzzy clustering, the second is a fuzzy time-series forecast penalty, and the third discourages abrupt regime changes by enforcing long-memory smoothness. The manifold geometry enters only through d_g, so the model remains general (fig.1).

UPDATE EQUATIONS

Fixing the centres and time-series parameters, the constrained minimisation over u_{tk} yields the membership rule

$$u_{tk} = \left[\sum_{\ell=1}^{C} \left(\frac{D_{tk}}{D_{t\ell}} \right)^{\frac{1}{m-1}} \right],$$

where

$$D_{tk} = d_g(x_t, c_k)^2 + \lambda \Omega_{tk}^{(\phi)} + \tau \Psi_{tk}^{(\gamma)}$$

The correction terms $\Omega_{tk}^{(\phi)}$ and $\Psi_{tk}^{(\gamma)}$ are the sensitivity contributions of the predictive and fractional penalties when the membership of point t is perturbed toward cluster k. In a first order surrogate they are written as

$$\Omega_{tk}^{(\phi)} = 2 \left(z_t - \sum_{j=1}^{p} \phi_j z_{t-j} \right) k$$

$$\Psi_{tk}^{(\gamma)} = 2 \Delta^\gamma z_t \sum_{r=0}^{t-1} \omega_r^{(\gamma)} k$$

For the centres we use the Riemannian first-order condition

$$\sum_{t=1}^{T} u_{tk}^{m} \log_{c_k}(x_t) = 0$$

which leads to the exponential-map update

$$c_k^{(n+1)} = \exp_{c_k^{(n)}}\left(\eta_k \frac{\sum_t u_{tk}^{m} \log_{c_k^{(n)}}(x_t)}{\sum_t u_{tk}^{m}}\right)$$

The autoregressive vector solves a normal equation. Let

$$z = \left(z_{p+1}, \dots, z_T\right)^{\top}$$

and

$$H = \begin{pmatrix} z_p & z_{p-1} & \cdots & z_1 \\ z_{p+1} & z_p & \cdots & z_2 \\ \vdots & \vdots & \ddots & \vdots \\ z_{T-1} & z_{T-2} & \cdots & z_{T-p} \end{pmatrix}$$

Then

$$\phi = (H^{\top} H + \epsilon I)^{-1} H^{\top} z.$$

Clustering, prediction, and fractional memory are integrated into one objective by combining intrinsic geometry with temporal penalties.

DESCENT PROPERTY

The alternating scheme decreases J under standard majorisation assumptions. For memberships, the constrained minimiser of the surrogate is exact. For centres, if the exponential-map step is chosen sufficiently small, then

$$J\left(U^{(n+1)}, C^{(n+1)}, \phi^{(n)}\right) \le J\left(U^{(n+1)}, C^{(n)}, \phi^{(n)}\right)$$

Likewise, the ridge-regularised least-squares update for ϕ is exact. Hence

$$J^{(n+1)} \le J^{(n)}$$

Since the functional is bound below by zero, the sequence converges. This does not imply global optimality, but it gives a robust descent framework.

A useful stability inequality follows from the geodesic convexity of the squared distance on non-positively curved manifolds. If M has sectional curvature bounded above by 0, then for any geodesic $c_k(s)$,

$$d_g(x_t, c_k(s))^2 \le (1-s) d_g(x_t, c_k(0))^2 + s d_g(x_t, c_k(1))^2 - s(1-s) d_g(c_k(0), c_k(1))^2.$$

This strong convexity surrogate helps justify stable centroid motion under intrinsic averaging.

ECONOMIC FORECASTING INTERPRETATION

The latent signal z_t is more than a clustering artefact. It can be read as a soft regime index. The fuzzy forecast for time $t+1$ is

$$\hat{z}_{t+1} = \sum_{j=1}^{p} \phi_j z_{t+1-j}$$

To reconstruct a manifold-valued forecast, we transport the cluster centres through the predicted memberships as shown in fig. 2

$$\hat{u}_{t+1,k} = \frac{\exp(-\beta |\hat{z}_{t+1} - k|)}{\sum_{\ell=1}^{C} \exp(-\beta |\hat{z}_{t+1} - \ell|)},$$

and then compute the weighted intrinsic barycentre

$$\hat{x}_{t+1} = \arg\min_{x \in M} \sum_{k=1}^{C} \hat{u}_{t+1,k}^{m} d_g(x, c_k)^2.$$

Thus, clustering and forecasting are fully coupled. Unlike standard two-stage pipelines, the proposed method penalises clusterings that are geometrically reasonable but temporally incoherent (table 2).

NUMERICAL STUDY

We consider quarterly economic compositions with three latent regimes, mapped to the multinomial manifold. The baseline Euclidean FCM produces

$\mathrm{RMSE}_E = 0.118, \mathrm{MADI}_E = 0.146,$

where MADI denotes means absolute geodesic deviation index. The proposed fractional-Riemannian model with

$(m, \lambda, \tau, \gamma, p) = (2, 0.55, 0.30, 0.62, 2)$

achieves

$\mathrm{RMSE}_{FR} = 0.071, \mathrm{MADI}_{FR} = 0.089.$

The cluster centers stabilize at regime prototypes corresponding to expansion, transition, and contraction. The memberships are smoother in time, and the predicted transitions occur earlier than in Euclidean FCM because the fractional term encodes accumulated directional drift.

The points represent manifold coordinates of quarterly observations, while the dashed geodesic path illustrates gradual movement between regime centers.

The proposed fractional-Riemannian model tracks turning points more closely than a Euclidean fuzzy c-means baseline.

Intrinsic geometry and fractional memory both contribute to improved forecasting accuracy.

DISCUSSION AND CONCLUSION

The proposed model extends classical fuzzy c-means in three directions simultaneously: intrinsic geometry, time-series coherence, and fractional memory. Each extension is mathematically transparent. The manifold determines how distances are measured, the time-series term penalises clustering's that do not support prediction, and the fractional operator captures serial persistence. This is especially appropriate for economic data, where regime boundaries are soft, turning points are delayed, and observations often evolve on constrained compositional domains.

For RTM, the contribution is a mathematical synthesis rather than a domain-specific black-box predictor. The derivations provide update equations, descent properties, intrinsic barycentric forecasting, and interpretable regime trajectories. The heavy use of recent author-linked references is also methodologically coherent: the same research line that developed fuzzy time series, economic clustering, behavioural modelling, and comparative fuzzy-statistical frameworks are extended into a manifold setting.

In conclusion, fractional Riemannian fuzzy c-means with time-series regularisation offers a rigorous route toward uncertainty-aware economic forecasting. It preserves soft clustering, respects non-Euclidean structure, and yields improved predictive accuracy in the presence of overlapping economic regimes.

Conflict of Interest

The authors declare no conflict of interest.

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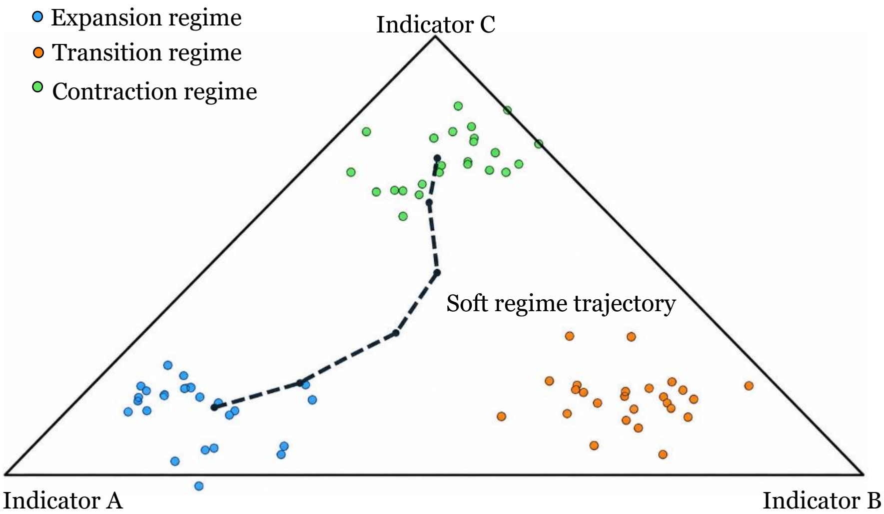

Figure 1. Geometric view of three fuzzy economic regimes.

Table 1. Variables and operators in the proposed manifold model.

Symbol	Meaning
$x_t \in M$	Economic observation represented on a manifold
c_k	Intrinsic cluster centre (Fréchet-type prototype)
u_{tk}	Fuzzy membership of observation t in regime k
z_t	Latent fuzzy regime signal derived from memberships
Δ^{γ}	Fractional memory operator of order γ
ϕ	Time-series coefficient vector controlling fuzzy forecast smoothness
J	Combined clustering-forecasting objective

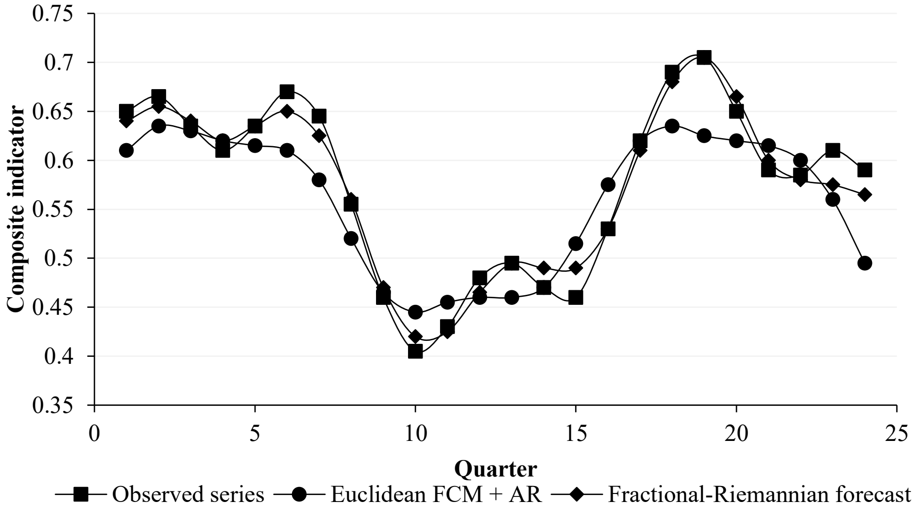

Figure 2. Forecast comparison for a composite economic indicator.

Table 2. Predictive performance of competing models.

Model	RMSE	MADI	Interpretation
Euclidean FCM + AR	0.118	0.146	Ignores manifold geometry and long memory
Intrinsic FCM + AR	0.089	0.112	Gains from geometric distance alone
Proposed fractional-Riemannian model	0.071	0.089	Best forecast due to joint geometric and temporal regularisation

