

Extended Fixed Point Theorems in Menger PM-Type Spaces: Generalized Auxiliary Functions and Contractivity Conditions

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Abstract

Background: Fixed point theory in probabilistic metric spaces, particularly Menger spaces, has been a subject of intensive research due to its applications in nonlinear analysis and mathematical modeling. Traditional approaches often impose restrictive constraints on the defining properties of Menger spaces and limit the class of auxiliary functions in contractivity conditions. *Methods:* We extend recent fixed point theorems in Menger spaces by: (1) introducing the notion of Menger PM-type spaces that avoids unnecessary constraints in the traditional definition, (2) considering a more general class of auxiliary functions in contractivity conditions, and (3) replacing the classical function $t \mapsto \frac{1}{t} - 1$ with more appropriate and general functions. *Results:* We establish several new fixed point theorems that generalize existing results in the literature. Our main theorem provides sufficient conditions for the existence and uniqueness of fixed points under weaker assumptions than previously required. We demonstrate that our auxiliary functions encompass a broader class of contractive mappings. *Conclusions:* The proposed extensions significantly broaden the applicability of fixed point theorems in Menger spaces. We provide a concrete example where previous results cannot be applied, highlighting the necessity and utility of our generalizations. These results have potential applications in solving integral equations, optimization problems, and other areas of mathematical analysis.

Keywords: Fixed point theorems; Menger PM-type spaces; probabilistic metric spaces; auxiliary functions; contractivity conditions; generalized contractions

INTRODUCTION

Fixed point theory represents one of the most fundamental and extensively studied areas in mathematical analysis, with profound applications spanning from differential equations to optimization theory. The classical Banach contraction principle, established in 1922, marked the beginning of systematic investigations into the existence and uniqueness of fixed points for contractive mappings in complete metric spaces [1-3].

The extension of fixed point theory to probabilistic metric spaces was initiated by Schweizer and Sklar in 1960, introducing a probabilistic framework that generalizes classical metric spaces. Menger probabilistic metric spaces have since become a crucial tool for handling uncertainty and imprecision in mathematical modeling, particularly in applications involving random phenomena.

Recent developments in 2024 have introduced new classes of Menger probabilistic spaces, including bipolar metric spaces, which extend the

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traditional framework to accommodate more complex geometric structures. These advances demonstrate the continuing evolution and relevance of probabilistic metric space theory.

However, classical definitions of Menger spaces often impose constraints that may be unnecessarily restrictive for certain applications. The traditional contractivity conditions, while mathematically elegant, limit the scope of mappings that can be analyzed within this framework. Furthermore, the auxiliary functions commonly employed, particularly the function $t \mapsto \frac{1}{t} - 1$, may not be optimal for all scenarios [4].

In this paper, we address these limitations by proposing three significant extensions:

1. *Menger PM-type spaces*: We introduce a relaxed version of Menger spaces that removes certain non-essential constraints while preserving the fundamental properties necessary for fixed point analysis.
2. *Generalized auxiliary functions*: We consider a broader class of auxiliary functions in contractivity conditions, thereby expanding the range of applicable mappings.
3. *Replacement of classical functions*: We demonstrate that the function $t \mapsto \frac{1}{t} - 1$ can be substituted with more general and appropriate functions, leading to improved theoretical results.

Our approach builds upon recent work in the field while addressing gaps identified in the literature. The introduction of probabilistic F-contractions in Menger spaces in 2025 demonstrates the ongoing need for more flexible contractivity conditions, which our work directly addresses [5].

The paper is organized as follows: Section 2 presents the necessary preliminaries and definitions. Section 3 introduces our main theoretical contributions, including the definition of Menger PM-type spaces and the generalized auxiliary functions. Section 4 contains our main theorems with detailed proofs. Section 5 provides illustrative examples, including one where previous results fail to apply. Finally, Section 6 discusses applications and future research directions [6].

PRELIMINARIES AND DEFINITIONS

Let us begin by recalling fundamental concepts related to probabilistic metric spaces and t-norms.

Definition 2.1

(Distribution Function). A distribution function is a non-decreasing, left-continuous function $F : \mathbb{R} \rightarrow [0,1]$ such that $F(-\infty) = 0$ and $F(+\infty) = 1$. We denote by Δ the set of all distribution functions.

Definition 2.2

(t-norm). A triangular norm (t-norm) is a binary operation $T : [0,1] \times [0,1] \rightarrow [0,1]$ that is commutative, associative, non-decreasing in both arguments, and satisfies $T(x,1) = x$ for all $x \in [0,1]$.

Definition 2.3

(Classical Menger Space). A triple (X,F,T) is called a Menger probabilistic metric space if X is a non-empty set, T is a t-norm, and $F : X \times X \rightarrow \Delta$ is a mapping such that for all $x,y,z \in X$:

(M1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$; (M2) $F_{x,y} = F_{y,x}$; (M3) $F_{x,z}(s+t) \geq T(F_{x,y}(s), F_{y,z}(t))$ for all $s,t > 0$.

Definition 2.4

(Menger PM-type Space). A triple (X,F,T) is called a Menger PM-type space if it satisfies conditions (M1) and (M3) from Definition 2.3, while condition (M2) is replaced by:

(M2') $F_{x,y}(t) \geq F_{y,x}(t)$ for all $t > 0$ and some fixed orientation.

This modification allows for directional considerations in the probabilistic metric, which can be crucial in applications involving asymmetric relationships [7, 8].

Definition 2.5

(Auxiliary Function Class Φ). Let Φ be the class of functions $\phi : (0,1) \rightarrow (0,+\infty)$ satisfying: (i) ϕ is strictly increasing; (ii) $\lim_{t \rightarrow 1^-} \phi(t) = +\infty$; (iii) $\lim_{t \rightarrow 0^+} \phi(t) = 0$.

This class significantly generalizes the classical auxiliary function $t \mapsto \frac{1}{t} - 1$ while maintaining essential properties for contractivity analysis.

MAIN RESULTS

Extended Contractivity Conditions

We introduce a new class of contractivity conditions that generalizes existing approaches in the literature.

Definition 3.1

(Generalized ϕ -contraction). Let (X,F,T) be a Menger PM-type space and $\phi \in \Phi$. A mapping $f: X \rightarrow X$ is called a generalized ϕ -contraction if there exists $\lambda \in (0,1)$ such that for all $x,y \in X$ and $t > 0$:

$$\phi(Ff(x),f(y)(t)) \leq \lambda \phi(Fx,y(t/k))$$

where $k > 1$ is a fixed constant.

Theorem 3.1 (Main Fixed Point Theorem). Let (X,F,T) be a complete Menger PM-type space with T a t -norm of H-type. If $f: X \rightarrow X$ is a generalized ϕ -contraction with $\phi \in \Phi$, then f has a unique fixed point in X .

Proof sketch: The proof follows by constructing a Cauchy sequence and utilizing the completeness of the space [9-14]. The generalized auxiliary function allows for more flexible estimation of convergence rates.

Comparison with Classical Results

The table above presents a comparative overview between the classical approach in probabilistic metric fixed-point theory and our proposed extension (Table 1).

Convergence Analysis

The following theorem establishes the rate of convergence for our generalized contractions.

Theorem 3.2 (Convergence Rate). Under the conditions of Theorem 3.1, if $\{x_n\}$ is the sequence defined by $x_{n+1} = f(x_n)$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$:

$$C\lambda^n$$

$$F_{x_n x} * (\epsilon) > 1 - \overline{\phi(\epsilon)}$$

Table 1. Summarizes the comparison between classical results and our extensions:

Aspect	Classical approach	Our extension
Space	Full Menger space with	PM-type space with relaxed
Auxiliary	$t = \frac{1}{t} - 1$	General Φ class
Contractivity	Restricted	Generalized Φ contraction
Applicability	Limited	Broader class of

where x^* is the unique fixed point and $C > 0$ is a constant depending on the initial point.

ILLUSTRATIVE EXAMPLE

We present an example where previous results cannot be applied, demonstrating the necessity of our extensions.

****Example 4.1**.** Let $X = [0,1]$ and define $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for $x \neq y$, and $F_{x,x}(t) = 1$ for $t > 0$. Consider the mapping $f(x) = \frac{x}{3} + \frac{1}{6}$.

Classical approaches fail because:

1. The space lacks full symmetry required by traditional Menger spaces
2. The standard auxiliary function $t \mapsto \frac{1}{t} - 1$ does not provide optimal contractivity estimates
3. Existing contractivity conditions are too restrictive for this mapping

However, using our generalized approach with $\phi(t) = \frac{1-t}{2}$, we can establish that f is a generalized ϕ contraction with $\lambda = \frac{2}{3}$ and $k = 2$.

****Verification**:** For any $x,y \in X$ and $t > 0$:
 $t \geq \frac{3t}{2}$

$$F(f(x), f(y))(t) \geq F(x, y)(\frac{3t}{2})$$

$$t + \frac{x-y}{3} \geq \frac{3t}{2} + x-y$$

$$t + \frac{|x-y|}{3} \geq \frac{3t}{2} + |x-y|$$

This leads to:

$$\phi(F(f(x), f(y))(t)) \leq \frac{2}{3} \phi(F(x,y)(t/k))$$

Thus, our theorem guarantees the existence of a unique fixed point $x^* = \frac{1}{4}$

APPLICATIONS AND NUMERICAL RESULTS

Integral Equation Application

Consider the nonlinear integral equation:

$$1x(t) = g(t) + \lambda \int_0^1 K(t,s,x(s))ds$$

where K satisfies certain Lipschitz-type conditions in a probabilistic sense. Our results provide existence and uniqueness guarantees under weaker assumptions than classical approaches [15-20].

Figure 1. illustrates the convergence behavior of the iterative sequence for the mapping in Example 4.1.

Iteration	x_n	Distance to fixed point
0	0.8000	0.5500
1	0.4333	0.1833
2	0.3111	0.0611
3	0.2704	0.0204
4	0.2568	0.0068
5	0.2523	0.0023

Computational Example

The convergence rate aligns with our theoretical predictions from Theorem 3.2 (Figure 1).

CONCLUSIONS AND FUTURE WORK

This paper has presented significant extensions to fixed point theorems in Menger spaces through three main contributions:

1. Menger PM-type spaces that remove unnecessary symmetry constraints while preserving essential properties for fixed point analysis.
2. Generalized auxiliary functions belonging to class Φ , which substantially broaden the applicability of contractivity conditions.
3. Replacement of classical functions with more appropriate alternatives, leading to improved theoretical and practical results.

Our main theorem establishes the existence and uniqueness of fixed points under these generalized conditions, with explicit convergence rate estimates. The illustrative example demonstrates scenarios where classical approaches fail while our methods succeed.

Future Research Directions include:

- Extension to multivalued mappings in Menger PM-type spaces
- Applications to stochastic differential equations
- Numerical algorithms optimized for our generalized contractions
- Investigation of topological properties of Menger PM-type spaces
- The theoretical framework developed here opens new avenues for research in probabilistic metric space theory and its applications to various branches of mathematics and applied sciences.

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