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Differential Model of Wave Vibration on Strings and Rods

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Abstract

Vibration, as a physical phenomenon, plays a multifaceted role in both natural and engineered systems. Beyond its essential contribution to communication through sound and speech, it finds critical application in various technological domains. For instance, in the design of stringed musical instruments such as violins and guitars, precise manipulation of string vibrations defines tonal quality and musical resonance. Similarly, metal rods in tuning forks exhibit vibrational properties that are finely tuned to produce specific frequencies. The mathematical formulation of vibration—primarily using the one-dimensional wave equation—enables accurate modeling of oscillations in both continuous and discrete systems. This provides the foundation for predictive simulations in physics and engineering. In civil and structural engineering, understanding and controlling vibration is crucial for the integrity and safety of buildings, bridges, and towers. Natural phenomena like seismic activities or man-made sources like vehicular motion on highways can induce resonant frequencies that compromise structural stability. Engineers apply finite element methods and dynamic analysis tools to evaluate modal frequencies and optimize structural response. Mitigation techniques such as base isolators, tuned mass dampers, and viscous fluid dampers are deployed strategically to absorb and dissipate kinetic energy from vibrations. These mechanisms not only reduce amplitude but also prevent structural fatigue, ensuring long-term durability. Consequently, vibration is not merely a physical event but a complex interaction of energy, material behavior, and dynamic stability that commands significant interdisciplinary attention

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Keywords: Mathematical modelling, One-dimensional wave differential equation, Trajectory analysis, Damping, Shock absorbers, Structural analysis and design

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INTRODUCTION

Transverse Waves and Harmonics

Waves triggered on a string are transverse waves. The displacements of oscillations of the waves are mutually orthogonal to the direction of the motion of wave propagation. The speed of travel of waves on a spring is dependent on the tension of developed in the string and the mass density of distribution of the string. The speed of wave, v, wavelength of vibration, λ and frequency of wave oscillation, f are related by the relationship $v = f\lambda$. In a situation where a string is pinned or fixed or restraint (nodes) at both ends, they vibrate at some least frequency defined as the fundamental frequency $(f_1)$of that string or rod. This frequency is the simplest frequency of that string with a peak displacement called the anti-node of one harmonic of the string or rod. In situations where a string vibrates with two or more anti-nodes, the attainable frequencies or harmonics are multiples of the fundamental frequency of the said string $(f_n = n \times f_1)$. n is the number of anti-nodes of vibration of the string or rod. The wavelength of string pinned or fixed of nth harmonics is twice the length of the string divided by the number of anti-nodes, n $(i.e.\ \lambda_n = 2L/n)$ [1-8].

The power of the transmission of wave energy with time is directly proportional to the product of the square of the square of the frequency of oscillation and the amplitude of vibration of the element $(i.e. P\alpha f^2 A^2)$. The general equation of waves on string is $h(x,t) = Ae^{-\xi t} sin(\omega t - kx + \phi)$, where A is the amplitude of wave, $k = 2\pi/\lambda$ is the wave number. $\phi = \omega_n \Delta\tau$ is the phase constant which defines the initial condition of the wave due to the delay in time, $\Delta\tau$ between the instant at perturbing and at its peak deflection in vibration. $\omega = 2\pi f$ is the angular frequency of wave and f is its frequency. $h(x,t)$ is the deflection of the string from its equilibrium position at any instant in time and space. $\xi = \frac{b}{2\omega_n M}$ is the damping ratio, ω_n is the natural frequency, b is the damping coefficient and M is the mass of the string. $e^{-\xi t}$ is the exponential decay factor due to damping effects [1-4].

Damping and Principle of Superimposition

Damping in string is a phenomenon that relinquishes and dissipates the mechanical energy of the vibratory system gradually due to air resistance or the medium, say water, internal resistance within the material and frictional forces and consequently depreciates the amplitude of oscillation. When multiple waves are propagated, they interfere with each other and are superimposed. In a case where two waves travel and converge in phase (crest to crest or trough to trough of waves), they amplify their resultant effect. Such phenomenon is known as Constructive Interference. In a case where two waves travel out of phase (crest to trough of waves) and converge, they cancel their resultant effect. Such phenomenon is known as Destructive Interference. These gave rise to the principle of superposition which states that when multiple waves interact, their resultant displacement is the algebraic sum of the individual displacements at any instant in time and space [1-8].

MATERIALS AND METHODS

Wave Vibrations on Strings and Rods

Consider displacement caused by excitation of a string with reference to an inertia (fixed) reference frame (O_{XY}) vibration in a curved mode shape as shown in Figure 1.

Figure 1. Curved trajectory of mass vibration of string in motion

$h(x,t) =$ Vertical displacement of string from equilibrium position at time, t
$\lambda =$ Mass per unit length of string

Consider an elemental mass strip of (dx) of the displaced uniform string in Figure 2.

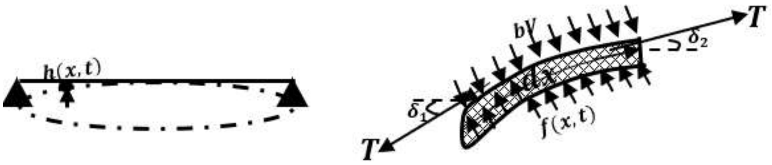

Figure 2. Forces on an elemental strip of the vibrating string

Elemental mass, $dm = \lambda dx$ (1)
$bV =$ Damping force per unit length of elemental strip

$f(x,t) =$ Excitation force per unit length of string or engineering material

Assume a small twist deflection of vibration motion

$$tan\delta_1 \approx sin\delta_1 \approx \delta_1 = \frac{\delta h}{\delta x} \quad (2)$$

$$tan\delta_2 \approx sin\delta_2 \approx \delta_2 = \frac{\delta h}{\delta x} + \frac{\delta}{\delta x}\left(\frac{\delta h}{\delta x}\right)dx = \frac{\delta h}{\delta x} + \frac{\delta^2 h}{\delta x^2}dx \quad (3)$$

Force balance on the elemental strip dx:

$$\sum F_v = T\left(\frac{\delta h}{\delta x} + \frac{\delta^2 h}{\delta x^2}dx\right) - T\left(\frac{\delta h}{\delta x}\right) + f(x,t)dx - bVdx = (dm)\ddot{y} \quad (4)$$

$$\rightarrow T\left(\frac{\delta h}{\delta x} + \frac{\delta^2 h}{\delta x^2}dx\right) - T\left(\frac{\delta h}{\delta x}\right) + f(x,t)dx - bVdx = \lambda\ddot{y}dx$$

$$T\frac{\delta^2 h}{\delta x^2}dx + f(x,t)dx - bVdx = \lambda\frac{\delta^2 h}{\delta t^2}dx \quad (5)$$

For natural frequency and mode shapes, neglecting $f(x,t)dx - bVdx$

$$T\frac{\delta^2 h}{\delta x^2}dx = \lambda\frac{\delta^2 h}{\delta t^2}dx \quad (6)$$

$$\frac{\delta^2 h}{\delta x^2} = \frac{\lambda}{T}\frac{\delta^2 h}{\delta t^2}$$

$$\frac{\delta^2 h}{\delta x^2} = \frac{1}{T/\lambda}\frac{\delta^2 h}{\delta t^2} \quad (7)$$

According to dimensional analysis:

$$\frac{T}{\lambda} = \frac{N}{kg/m} = m^2 s^{-2} = [c^2] = Speed\ of\ wave\ in\ the\ medium$$

$$\frac{\delta^2 h}{\delta x^2} = \frac{1}{c^2}\frac{\delta^2 h}{\delta t^2}$$

Hence, the one dimensional wave equation of string vibration is given in eqn. (8)

$$\frac{\delta^2 h}{\delta x^2} = \frac{1}{c^2}\frac{\delta^2 h}{\delta t^2} \quad (8)$$

$$c^2 = \frac{T}{\lambda} = \frac{T}{m}$$

Phase velocity, $c = \sqrt{\frac{T}{\lambda}} = \sqrt{\frac{T}{m}}$

$$c = \sqrt{\frac{T}{m}} \tag{9}$$

Where, $\lambda, m = mass\ per\ unit\ length\ of\ string$

To solve the partial differential equation of eqn. (8):

$$\frac{\delta^2 h}{\delta x^2} = \frac{1}{c^2}\frac{\delta^2 h}{\delta t^2}$$
$$h = h(x,t)$$

Assume solution: $h(x,t) = X(x) \cdot T(t)$

$$\frac{\delta h}{\delta x} = T\frac{dX}{dx}$$
$$\frac{\delta^2 h}{\delta x^2} = T\frac{d^2 X}{dx^2}$$
$$\frac{\delta h}{\delta t} = X\frac{dT}{dt}$$
$$\frac{\delta^2 h}{\delta t^2} = X\frac{d^2 T}{dt^2}$$

Plug in $\frac{\delta^2 h}{\delta x^2} = T\frac{d^2X}{dx^2}$ and $\frac{\delta^2 h}{\delta t^2} = X\frac{d^2T}{dt^2}$ into the wave model in eqn. (8), we have:

$$T\frac{d^2X}{dx^2} = \frac{X}{c^2}\frac{d^2T}{dt^2}$$

Separating the variables:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = constant, k$$
$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = k$$
$$\frac{1}{X}\frac{d^2X}{dx^2} = k$$
$$\frac{d^2X}{dx^2} - kX = 0 \tag{10}$$

Assume: $X(x) = Ae^{sx}$

Plug in $X(x) = Ae^{sx}$ into eqn. (10)

$$s^2 - k = 0$$
$$s = \pm\sqrt{k}$$
$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = k$$
$$\frac{d^2T}{dt^2} - kc^2T = 0 \tag{11}$$

Assume: $T(t) = Be^{st}$
Plug in $T(t) = Be^{st}$ into eqn. (11)
$s^2 - kc^2 = 0$
$s = \pm c\sqrt{k}$

Hence,
$$s = \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} \pm\sqrt{k} \\ \pm c\sqrt{k} \end{Bmatrix} \quad (12)$$

The following three possible cases of solutions to the wave equation can be analyzed using mathematical approaches of differential equations:
Case 1: $k > 0$
Case 2: $k = 0$
Case 3: $k < 0$

CONCLUSION

Sound, whether music or noise, speech for communication, cannot exist without the existence of vibration. Vibration, when is undesirable can cause catastrophic effects, such involve the damage of structures, loss of lives, assets, valuables, hearing deficiencies, etc. This paper explained the concept of vibration on strings and rods in mathematical derivations which are applicable in musical instrumentation and real life physics of such media. Waves generation and propagation of vibration on strings and rods were explained in physics with mathematics principles and modelled in time and space in one-dimensional differential wave equation. Structural engineers analyzes vibration caused by traffic, wind and earthquakes in structural elements and designed to withstand such loadings, deter failure and suffice such calamitous and appalling effects. Dampers and shock absorbers are useful to ameliorate and palliate the impact of unwanted vibration in buildings and enhance the longevity, sustaining the lifespan of the structures.

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