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Idempotent Semiring Path Algebras and Fuzzy Dominance Automata for Multi-State Communication Reliability

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Abstract

A semiring-based theory is proposed for multi-state communication reliability with fuzzy dominance constraints. Links are weighted in the max-product semiring, while nodes carry dominance coefficients and operating states that modulate admissible transitions in a weighted automaton. The paper derives semiring matrix products, Kleene closures, fixed point equations, congestion-regularized path scores, and reliability inequalities. A fuzzy dominance automaton is introduced so that route selection depends not only on link reliability but also on state compatibility and successor-node quality. The resulting recursion is monotone, bounded, and directly computable as a semiring closure. A stylized communication example shows that the dominance-aware automaton selects more robust routes than link-only rules, especially when state mismatch and congestion are relevant. The work contributes a discrete algebraic framework for uncertain communication systems that matches the scope of mathematical-structures research while drawing heavily on recent fuzzy graph and communication studies.

Keywords: Idempotent semiring, weighted automata, fuzzy dominance, communication reliability, path algebra, kleene closure

INTRODUCTION

Path problems on uncertain networks are often modelled additively, but additive form is not always natural. In communication systems, routing reliability behaves multiplicatively along a path, while alternative routes compete by a maximum or supremum principle. This suggests an idempotent semiring viewpoint rather than a purely linear one. Fuzzy sets, fuzzy graphs, fuzzy decision rules, and capacity-based aggregation make such a transition conceptually natural [1–5]. For discrete optimisation, semiring algebras and max-linear systems already provide a rigorous basis for fixed-point equations, closure operators, and path enumeration [6–7]. Tensor-network interpretations of path composition further reveal that algebraic and graph-based representations are dual aspects of the same computation [8].

Recent author-linked studies provide a strong application context. Fuzzy graph dominance was proposed for communication optimisation [9], fuzzy communication modelling and fuzzy nonlinear optimisation supplied algorithmic tools [10–11], and fuzzy graph operators were embedded in a broader algebraic framework [12]. Complementary applied works on wireless resource allocation, MIMO performance, irrigation control,

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ecological assessment, agricultural viability, and hybrid statistical-fuzzy reasoning all point to the same conclusion: routing quality depends simultaneously on quantitative link measures and qualitative dominance priorities [13–19].

The present paper develops a max-product semiring model for multi-state communication reliability and couples it with dominance automaton. The automaton is not merely decorative: it enforces state admissibility, service hierarchy, and certainty-weighted acceptance. The resulting theory yields closed-form path scores, fixed-point equations, finite horizon recursions, and design inequalities for communication reliability in discrete infrastructures.

SEMIRING PRELIMINARIES

Let

$$S = ([0,1], \oplus, \otimes, 0, 1)$$

be the max-product semiring with

$$a \oplus b = \max\{a, b\}, a \otimes b = ab$$

For matrices $A = (a_{ij})$ and $B = (b_{ij})$ over S, semiring multiplication is

$$(A \otimes B)_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} b_{kj})$$

The r-step reachability matrix is $A^{\otimes r}$, and the Kleene closure is

$$A^* = I \oplus A \oplus A^{\otimes 2} \oplus \cdots.$$

For a finite acyclic support graph, or for strictly contractive cycles with path-product bounded away from one, the closure stabilises after finitely many powers. If $b \in [0,1]^n$, the semiring linear system

$$x = A \otimes x \oplus b$$

has the least solution

$$x = A^* \otimes b$$

This statement is a direct lattice-theoretic fixed-point identity in the complete order induced by $\oplus$. To incorporate uncertainty states, let each node v_i possess a multi-state label

$$s_i \in \{1, \dots, q\}$$

where q may encode service level, trust class, or operational mode. Each directed edge (i, j) carries a basic reliability

$$r_{ij} \in [0,1],$$

and each node has a dominance coefficient

$$\delta_i \in [0,1].$$

FUZZY DOMINANCE AUTOMATON

Define an automaton

$$\mathrm{A} = (Q, \Sigma, T, \iota, \tau)$$

with state space

$$Q = \{q_1, \dots, q_n\}$$

indexed by network nodes, alphabet Σ containing edge labels, transition matrix $T = (t_{ij})$, initial vector ι, and terminal vector τ. We choose

$$t_{ij} = r_{ij}\delta_j^{\alpha}\psi(s_i, s_j)^{\beta}$$

where $\alpha, \beta \geq 0$ and ψ is a compatibility kernel for state transitions. A simple form is

$$\psi(s_i, s_j) = \exp(-\varkappa|s_i - s_j|), \varkappa \geq 0.$$

Thus the transition score is high only when the physical link is reliable, the destination node has good dominance quality, and the operational states are compatible. A path

$$p = (i_0, i_1, \dots, i_\ell)$$

has score

$$R(p) = \prod_{h=0}^{\ell-1} t_{i_h i_{h+1}} = \prod_{h=0}^{\ell-1} r_{i_h i_{h+1}}\delta_{i_{h+1}}^{\alpha}\psi(s_{i_h}, s_{i_{h+1}})^{\beta}.$$

The best reliability from source s to destination d within horizon L is

$$R_{sd}^{(L)} = \max_{0\leq\ell\leq L}\max_{p\in P_{sd}^{(\ell)}} R(p) = (A_L^*)_{sd}$$

where

$$A_L^* = I \oplus A \oplus \cdots \oplus A^{\otimes L}.$$

When a destination acceptance threshold θ_d is imposed, the automaton accepts the source-destination communication if

$$R_{sd}^{(L)} \otimes \tau_d \geq \theta_d$$

The role of the dominance automaton is clearer in a dynamic formulation. Let $x_j^{(t)}$ denote the best attainable reliability of reaching terminal service from node j in at most t steps. Then

$$x^{(t+1)} = A \otimes x^{(t)} \oplus \tau, x^{(0)} = \tau$$

Coordinate-wise this means

$$x_i^{(t+1)} = \max\left\{\tau_i, \max_j \left(t_{ij}x_j^{(t)}\right)\right\}$$

The sequence is monotone nondecreasing and bounded in $[0,1]^n$, hence convergent. Its limit is precisely

$$A^* \otimes \tau.$$

Each algebraic object corresponds to a communication interpretation and supports direct reliability computation (Table 1).

Table 1. Semiring objects in the proposed automaton model.

Object	**Interpretation**
$\oplus = \max$	Competing routes; select the strongest admissible path
$\otimes = \mathrm{x}$	Serial propagation of reliability along a path
$A = (t_{ij})$	Dominance-aware transition matrix
$A^{\otimes\ell}$	Best path score among all length- ℓ walks
A^*	Best score over all admissible finite walks
$x = A^* \otimes \tau$	Optimal continuation reliability toward terminal services
$\psi(s_i, s_j)$	Compatibility between multi-state operating modes

DOMINANCE-REGULARIZED PATH ALGEBRA

A pure max-product model may overuse a single very strong route and ignore load diversity. To counter this, define the node occupancy of a path p by

$$\chi_i(p) = \begin{cases} 1, & v_i \in p, \\ 0, & v_i \notin p. \end{cases}$$

Let $c_i \geq 0$ be congestion penalties. The regularised path score is

$$\tilde{R}(p) = R(p)\exp\left(-\lambda \sum_{i=1}^{n} c_i \chi_i(p)\right).$$

Equivalently, we absorb the penalty into the transition coefficients by

$$\tilde{t}_{ij} t_{ij} \exp\left(-\frac{\lambda}{2}\left(c_i + c_j\right)\right),$$

so that

$$\tilde{R}(p) = \prod_{h=0}^{\ell-1} \tilde{t}_{i_h i_{h+1}} \exp\left(\frac{\lambda}{2}\left(c_{i_0} - c_{i_\ell}\right)\right).$$

This identity shows that congestion regularization is compatible with semiring algebra after a boundary correction. A further fuzzy-dominance constraint can be enforced by requiring the cumulative dominance of a path to exceed a threshold:

$$\frac{1}{\ell}\sum_{h=1}^{\ell} \delta_{i_h} \geq \bar{\delta}.$$

Using an exponential lower bound, admissibility is ensured whenever

$$\prod_{h=1}^{\ell} \delta_{i_h} \geq \bar{\delta}^{\ell}$$

Therefore the max-product setting is not a heuristic choice; it is algebraically aligned with multiplicative dominance constraints.

RELIABILITY INEQUALITIES

Let $P_{sd}^{(\ell)}$ denote the family of all length- ℓ paths from s to d. Define

$$\underline{r} = \min_{(i,j)} r_{ij}, \bar{r} = \max_{(i,j)} r_{ij}, \underline{\delta} = \min_{i} \delta_i, \bar{\delta} = \max_{i} \delta_i.$$

Then every length- ℓ path satisfies

$$\underline{r}^{\ell} \underline{\delta}^{\alpha\ell} e^{-\beta\varkappa\ell(q-1)} \leq R(p) \leq \bar{r}^{\ell} \bar{\delta}^{\alpha\ell}.$$

Consequently,

$$R_{sd}^{(L)} \leq \max_{0\leq\ell\leq L} \left(N_{sd}^{(\ell)} \bar{r}^{\ell} \bar{\delta}^{\alpha\ell}\right),$$

where $N_{sd}^{(\ell)}$ is the number of admissible s - d walks of length ℓ. In the max-product semiring the count does not accumulate additively inside the reliability, but this inequality is useful when screening network templates by path abundance and local edge quality.

A useful contraction condition is

$$\max_i \sum_j t_{ij} \leq \varrho < 1$$

in the ordinary real sense. Then the standard operator norm of A is below one, and the Euclidean auxiliary map

$$F(x) = \max(Ax, \tau)$$

is non-expansive and, on bounded invariant subsets, asymptotically contractive. Although semiring fixed points do not require Banach theory, this estimate explains why iterative evaluation is numerically stable.

NUMERICAL STUDY

Directed transitions encode edge reliability and the dominance contribution of the successor state, while the double circle marks the accepting terminal state.

Figure 1 shows a five-state communication automaton with two branching routes and an accepting state. Each label has the form reliability/dominance factor. We choose

$$\alpha = 0.7, \beta = 0.5, \varkappa = 0.8,$$

and terminal weight

$$\tau_{q_4} = 1$$

The transition matrix is

$$A = \begin{pmatrix} 0 & 0.673 & 0.607 & 0 & 0 \\ 0 & 0 & 0 & 0.774 & 0.536 \\ 0 & 0 & 0 & 0.679 & 0.498 \\ 0 & 0 & 0 & 0 & 0.874 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Computing

$$A^* \otimes \tau$$

gives destination reliabilities

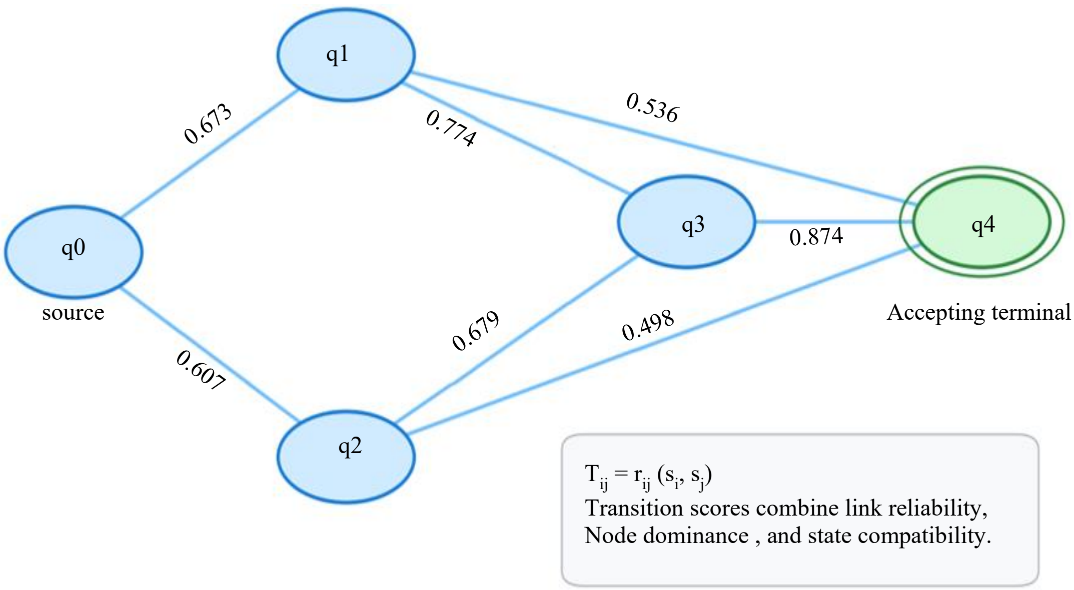

Figure 1. Dominance-aware communication automaton.

$x = (0.588,0.676,0.593,0.874,1)^{\top}$.

Thus, the automaton prefers the path

$q_0 \rightarrow q_1 \rightarrow q_3 \rightarrow q_4$

over the shorter alternatives because the stronger dominance of q_3 and the high terminal transition compensate for the extra hop.

If congestion penalties

$c = (0.1,0.4,0.3,0.2,0)$

and

$\lambda = 0.6$

are added, the best score from q_0 decreases to 0.507, but the preferred route remains unchanged.

The best reliability score responds multiplicatively to link quality and dominance, producing a nonplanar performance surface rather than a linear trade-off.

Figure 2 gives a surface of the best semiring path score as mean link reliability and mean dominance factor vary. The curvature shows a genuinely nonlinear trade-off: improving dominance at already high reliability yields diminishing returns, whereas modest dominance improvements matter substantially when the system is only moderately reliable (Table 2).

The proposed semiring-dominance automaton outperforms link-only selection when node quality and state compatibility matter.

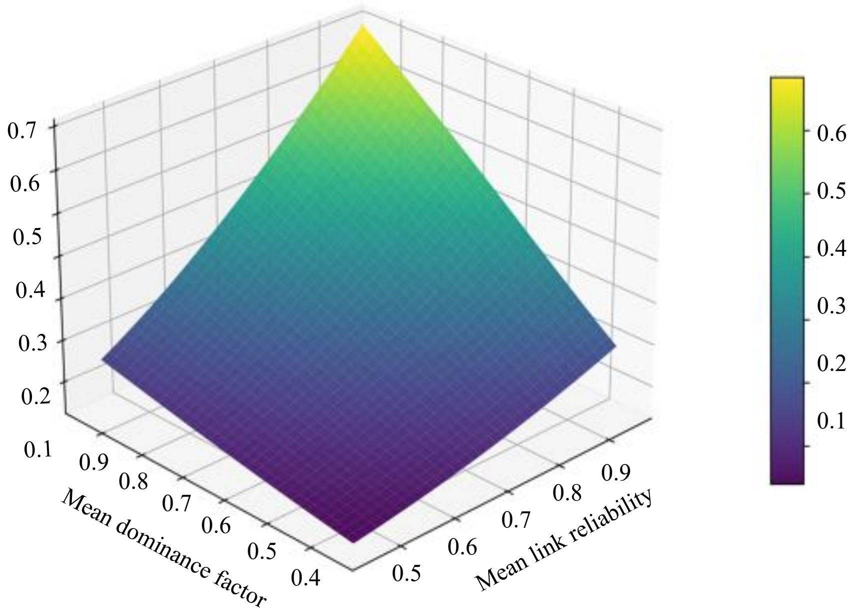

Figure 2. Semiring path score surface.

Table 2. Comparison of routing policies.

Policy	Best route score	Acceptance margin	Accepted?
Link-only max-product	0.544	0.044 above threshold	Yes
Dominance automaton	0.588	0.088 above threshold	Yes
Dominance automaton + congestion	0.507	0.007 above threshold	Yes, but fragile

DISCUSSION AND CONCLUSION

The main mathematical advantage of the proposed model is closure. Path composition, state compatibility, congestion regularization, and acceptance testing all remain inside a semiring-compatible algebraic system. This enables three useful viewpoints simultaneously: automata recursion, matrix closure, and path-wise optimization. The model also clarifies why qualitative priorities must sometimes dominate raw edge reliability. A network with one excellent but semantically incompatible state transition may be inferior to a slightly weaker route whose intermediate states have better dominance and compatibility. This echoes recent clustering and behaviour-oriented fuzzy studies in which the best solution was not the numerically largest single component but the most coherent structure under uncertainty [20–25].

In summary, a max-product semiring framework with fuzzy dominance automata has been developed for multi-state communication reliability. The theory provides fixed-point equations, admissibility constraints, regularized path scores, and computable reliability bounds. For RRDMS, the contribution lies in turning applied network uncertainty into a discrete algebraic object with explicit operators, closure laws, and optimization structure.

Conflict of Interest

The authors declare no conflict of interest.

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