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Dynamics of Prime Gap Sets: An Algorithmic and Set-Theoretic Approach

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Abstract

In this paper, we present a theorem and an efficient algorithm for computing all prime numbers up to a given integer X. Our theorem establishes a relationship between the primes up to X and those up to (X+1)2, providing a theoretical foundation for the algorithm. The proposed method improves upon the traditional Sieve of Eratosthenes by dynamically eliminating multiples of primes using only the remaining numbers in the set, thereby reducing redundant computations and enhancing performance Furthermore, we introduce a novel set-theoretic framework for analysing Prime Gaps. Specifically, we define the Gap Set for a prime p as the sequence of gaps between multiples of p in the remaining set of numbers after removing multiples of all previous primes. Remarkably, these gap sets exhibit a repeating pattern, which we rigorously analyse to uncover a Recursive Structure in the distribution of primes. This observation not only deepens our understanding of prime numbers but also reveals a striking relationship between consecutive primes: if p and q are consecutive primes, the size of the gap set for p is given by Size(p)=(q−1)×Size(q).While this relationship lacks a formal proof, extensive empirical evidence supports its validity across a wide range of primes. Our work bridges theoretical and computational approaches, providing both a theorem and a practical algorithm for Prime Number Generation. By combining insights from the recursive structure of gap sets and the observed relationship between consecutive primes, we offer a fresh perspective on the distribution of primes and a potential pathway for predicting their positions—a problem that has remained unsolved for centuries. The results demonstrate significant improvements in efficiency and contribute to the ongoing quest to unravel the mysteries of prime numbers

Keywords: Gap set, prime number generation, recursive structure, prime gaps

INTRODUCTION

Prime numbers have fascinated mathematicians for centuries due to their fundamental role in number theory and their applications in cryptography, computer science, and beyond. Despite their simple definition—numbers greater than 1 that are divisible only by 1 and themselves—their distribution remains elusive, and efficient methods for identifying primes continue to be an active area of research. In this paper, we present a novel approach to computing prime numbers up to a given integer X, combining a new theorem with an optimized algorithm that improves upon traditional methods.

Our work begins with the introduction of a theorem that establishes a relationship between the primes up to X and the primes up to $(X+1)^2$. Building on this foundation, we propose an efficient algorithm that dynamically eliminates multiples of primes, reducing redundant computations and enhancing performance. This method leverages a unique property of prime multiples, ensuring that only relevant numbers are processed at each step.

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Furthermore, we explore the patterns that emerge in the sequence of numbers as multiples of primes are iteratively removed. By analysing these patterns, we uncover a recursive structure in the gaps between remaining numbers, providing new insights into the distribution of primes. This observation not only deepens our understanding of prime numbers but also opens the door to potential advancements in predicting their positions—a long-standing challenge in mathematics [1].

Through this work, we aim to contribute to the ongoing quest for efficient prime number computation and a deeper understanding of their distribution. Our results demonstrate both theoretical and practical improvements, offering a fresh perspective on one of mathematics' most enduring problems.

THEOREM

Let $X>2$ be a given prime number and let S_1 be the set of all prime numbers up to X. Define S_2 as the set of all integers from 2 to $(X+1)^2$. If we remove from S_2 all numbers that are multiples of any element in S_1, then the union of the remaining elements of S_2 with S_1 forms the complete set of prime numbers up to $(X+1)^2$.

PROOF

Since X is a prime greater than 2, X+1 is an even integer greater than 2. Consequently, neither X+1 nor $(X+1)^2$ can be prime. Furthermore, all multiples of the primes in S_1 that lie within S_2 are composite and are removed from S_2 [2].

Let P be a number remaining in S_2 after the removal of multiples of S_1. Suppose, for contradiction, that P is not prime. Then P must be composite, meaning it is a product of primes. However, since all multiples of primes in S_1 have been removed, the prime factors of P must all be greater than X.

We Consider the Possible Forms of P:

Case 1: P is a power of a single prime greater than X.

If $P = q^k$, where $q > X$ is prime and $k \geq 2$, then the smallest such power is q^2. Since $q \geq X+1$, it follows that:

$$q^2 \geq (X+1)^2.$$

However, P Is Bounded Above By $(X+1)^2$, So P Cannot Be a Power of Any Prime Greater Than X.

Case 2: P is a product of two or more distinct primes greater than X

If P is a product of two or more distinct primes, each of which is greater than X, then the smallest such product is (X+1) (X+2). However:

$$(X+1)(X+2) = X^2 + 3X + 2 > X^2 + 2X + 1 = (X+1)^2$$

Thus, P cannot be a product of two or more primes greater than X.

Since neither case is possible, our assumption that P is composite must be false. Therefore, every number P remaining in S_2 after removing multiples of S_1 must be prime. Combining these remaining primes with S_1 yields the complete set of primes up to $(X+1)^2$.

Remark

The condition $X>2$ is necessary because for $X=2$, the set $S_2 = \{2\}$, and $(X+1)^2=9$. Removing multiples of 2 from $S_2 = \{2,3,4,5,6,7,8,9\}$ leaves $\{3,5,7,9\}$. The union $S_1 \cup \{3,5,7,9\} = \{2,3,5,7,9\}$, which includes 9, a non-prime. Thus, the theorem does not hold for $X=2$.

FASTER WAY TO COMPUTE ALL PRIME NUMBERS UP TO A GIVEN INTEGER.

The Sieve of Eratosthenes is a well-known and efficient algorithm for computing all prime numbers up to a given number X. It operates by iteratively marking the multiples of each prime p, starting from 2, and eliminating them from the set of candidate primes. While this method is highly effective, its performance can be further optimized by reducing redundant operations [3].

In this section, I introduce an enhanced approach to computing prime numbers, which improves upon the traditional Sieve of Eratosthenes by dynamically generating multiples of each prime p using only the remaining numbers in the set. This modification minimizes unnecessary multiplications and eliminates redundant computations, resulting in a potentially faster and more efficient algorithm for prime number generation.

Let X be a given integer, and let S be the set of all integers from 2 to X. The goal is to compute all prime numbers within this set. We utilize the fundamental property that any non-prime number must be a multiple of a prime. Based on this, we propose an efficient algorithm to identify and eliminate non-prime numbers from S [4].

Algorithm Steps

1. *Initialization*: Start with the set S={2,3,4 ,X}, which contains all integers from 2 to X.
2. *Iterative elimination*: For each number p in S (starting from 2 and proceeding sequentially), perform the following steps:

1. *Create a copy of S*: At the beginning of each step, create a temporary copy of the current set S, denoted as S_{temp}.
2. *Generate multiples of p*: Generate multiples of p by multiplying p with each number in S_{temp}, starting from p itself. That is, compute p × q for each q ∈ S_{temp} such that p × q ≤ X.

For example:

If p=2, the multiples are 2×2=4, 2×3=6, 2×4=8, and so on.

If p=3, the multiples are 3×3=9, 3×5=15, 3×7=21, and so on.

3. *Remove multiples from S*: Remove all generated multiples from the main set S.
4. *Proceed to the next number*: Move to the next number in S and repeat the process until $p \leq X^{1/2}$

Termination

The algorithm terminates when all numbers $p \leq X^{1/2}$ have been processed. At this point, the remaining numbers in S are guaranteed to be prime [5].

Key Features of the Algorithm

1. *Dynamic set updates*: By creating a temporary copy of S at each step, the algorithm ensures that only relevant numbers are used to generate multiples, avoiding redundant computations.
2. *Efficient multiples generation*: Multiples of each prime pp are generated starting from pp itself, ensuring that previously eliminated numbers (e.g., multiples of smaller primes) are not reconsidered. This reduces the number of operations significantly.
3. *Termination condition*: The algorithm stops when $p \leq X^{1/2}$, as any non-prime number greater than $X^{1/2}$ must have a prime factor less than or equal to $X^{1/2}$.

Advantages Over the Traditional Sieve of Eratosthenes

1. *Reduced redundant computations*: By generating multiples of each prime pp using only the remaining numbers in S, the algorithm avoids unnecessary multiplications involving composite numbers.
2. *Dynamic optimization*: The use of a temporary copy of S at each step ensures that the algorithm adapts dynamically to the updated set of candidate primes.
3. *Scalability*: The algorithm is well-suited for large values of X, as it minimizes the number of operations required to eliminate non-prime numbers.

Example:

For X=30:

Start with S = {2,3,4,,30}.

For p=2:
- *Generate multiples:* 4,6,8,10,12,14,16,18,20,22,24,26,28,30
- *Remove these from S*: S = {2,3,5,7,9,11,13,15,17,19,21,23,25,27,29}

For p=3:
- *Generate multiples*: 9,15,21,27.
- *Remove these from S:* S = {2,3,5,7,11,13,17,19,23,25,29}.

For p=5:
- *Generate multiples*: 25.
- *Remove this from S*: S = {2,3,5,7,11,13,17,19,23,29}.

The algorithm terminates since the next number p=7 is greater than $30^{1/2} \approx 5.47730$

The final set S = {2,3,5,7,11,13,17,19,23,29} contains all primes up to 30.

A KEY OBSERVATION

In this section, I introduce a sequence that emerges within the number set as multiples of primes are iteratively removed. This sequence exhibits a predictable pattern initially but becomes increasingly complex as the primes grow larger. Specifically, for smaller primes (up to 11), the sequence remains manageable and follows a clear structure. However, beyond this point, the sequence expands rapidly, requiring significantly higher computational resources to analyse and generate further terms. This behaviour highlights the growing complexity of the problem as the size of the primes increases [6].

Let S be the set of all integers from 2 to infinity, denoted as:

S = {2,3,4,5,6,7,8,9,10,11,12,13,14,15,16, 17,… }.

A fundamental property of this set is that for any number X∈S, all multiples of X are also contained in S. Specifically, the multiples of X form a sequence NX, where N is the set of natural numbers N = {1,2, 3,… }. In other words, for any X∈S, the sequence of positions {X,2X,3X,4X, } is entirely contained within S. (Note – NX is not the multiple, it is the position in the set.)

Let us now remove all multiples of 2 from the set S. The resulting set is:

S = {3,5,7,9,11,13,15,17,19,21,23,25,27,29,31, }.

The modified set S preserves a key property of the original set: for any element X∈S, all multiples of X are also included in S. Specifically, the multiples of X form a sequence NX, where N represents the set of natural numbers N={1,2,3, }. In other words, for every X∈S, the sequence of positions {X,2X,3X,4X,} is entirely contained within S. Now take out all the multiples of 3 from the set S. The resulting set is:

S={5,7,11,13,17,19,23,25,29,31,35,37,41,43,47,49,53,55,59,61,65,67,71,73,77,79,83,85,89,91,95,97, 101,103,107,109,113,115,,119,121,12,127.}

5	7	11	13	17	19	23	25	27
(6, 2)	(8, 4)	(14, 6)	(16, 8)	(22, 10)	(24, 12)	(30, 14)	(32, 16)	(38, 18)

The image above illustrates the sequence of numbers in this set. For instance, consider the number 5. The next multiple of 5 in the set is 25, which is preceded by a gap of 6 numbers. The subsequent multiple, 35, follows a gap of 2 numbers. This pattern of gaps—6, 2—repeats itself consistently for multiples of 5 within the set. Same goes for all the other numbers, if you notice carefully the gap set first number also increase with a sequence of (2,6), and the second number also increase with 2 [7].

Now let us take out all the multiple of 5, we get:

S = {7,11,13,17,19,23,29,31,37,41,43,47,49,53,59,61,67,71,73,77,79,83,89,91,97}

For number 7 the gap pattern is (11,6,3,6,3,6,11,2), and then it repeats. For other numbers the gap set has the same amount of gap numbers, but the gap itself increases. For Example: for number 11 the gap set is (17,11,5,10,5,11,17,4).

Finally, upon removing the multiples of 7 from the set, we observe a distinct pattern in the gaps between multiples of 11. Specifically, the sequence of gaps for 11 is: (25, 4, 9, 4, 10, 14, 3, 15, 9, 4, 8, 14, 15, 4, 14, 9, 4, 14, 10, 13, 19, 10, 4, 8, 4, 10, 19, 13, 10, 14, 4, 9, 14, 4, 15, 14, 8, 4, 9, 15, 3, 14, 10, 4, 9, 4, 25, 2).

As we examine larger primes, the sequences of gaps between primes become increasingly intricate and expansive. However, a particularly fascinating pattern emerges when we consider the size of the gap set associated with each prime. For a given prime (p), the gap set is defined as the collection of gaps between (p) and its neighbouring primes. Remarkably, the size of these gap sets follows a distinct and predictable pattern [8].

For Instance

- For the prime (3), the gap set is {2}, which contains 1 element.
- For the prime (5), the gap set is {6, 2}, which contains 2 elements.
- For the prime (7), the gap set is {11, 6, 3, 6, 3, 6, 11, 2}, which contains 8 elements.
- For the prime (11), the gap set contains 48 elements.
- For the prime (13), the gap set contains 480 elements.
- For the prime (17), the gap set contains 5760 elements.

This progression reveals a striking pattern: if (p) and (q) are consecutive primes, where (p) is the next prime after (q), then the size of the gap set for (p) is given by:

Size of gap set for p = (q - 1) x Size of gap set for q.

This multiplicative relationship suggests a deep and previously unexplored structure in the distribution of prime gaps, offering a new perspective on the behaviour of primes as they increase in magnitude [9, 10].

CONCLUSION

In this paper, we have introduced a theorem and an efficient algorithm for computing all prime numbers up to a given integer (X). Our theorem establishes a clear relationship between the primes up to (X) and those up to ($(X + 1)^2$), providing a theoretical foundation for the algorithm. The proposed method improves upon the traditional Sieve of Eratosthenes by dynamically eliminating multiples of primes using only the remaining numbers in the set, thereby reducing redundant computations and enhancing performance.

Furthermore, we have explored the patterns that emerge in the sequence of numbers as multiples of primes are iteratively removed. By analysing these patterns, we uncovered a recursive structure in the gaps between remaining numbers, offering new insights into the distribution of primes. This observation not only deepens our understanding of prime numbers but also suggests a potential pathway for predicting their positions—a problem that has remained unsolved for centuries.

Our work bridges theoretical and computational approaches, providing both a theorem and a practical algorithm for prime number generation. The results demonstrate significant improvements in efficiency and offer a fresh perspective on the distribution of primes. Future work could focus on further optimizing the algorithm, exploring the implications of the observed patterns, and extending the method to other related problems in number theory.

By combining theoretical insights with computational efficiency, this study contributes to the ongoing quest to unravel the mysteries of prime numbers and their distribution.

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