

# Flexural Vibration of Non-Uniform Beam on Bi-Parametric Foundation Under Harmonic Moving Load with Non-Classical Boundary Conditions

Jimoh A.<sup>1</sup>, Ajoge E.O.<sup>2\*</sup>, Ajiola D.I.<sup>3</sup>, Nicholas O.O.<sup>4</sup>, Adeyemi S.S.<sup>5</sup>, Abdulkabir L.A.<sup>6</sup>

## Abstract

*The dynamic response to harmonically varying moving loads of a non-uniform elastic beam resting on a bi-parametric foundation with non-classical boundary conditions, time-dependent in particular, is examined in this paper. Here, the beam is considered non-uniform, with the moment of inertia and the distributed mass varying continuously along the span  $L$  as functions of the position  $x$ . To guarantee a more stable structure and a reduced risk of resonance, the damping term is incorporated into the governing partial differential equations that describe the dynamical system. The effects of other structural parameters, such as foundation stiffness ( $K$ ), shear modulus ( $G$ ), axial force ( $N$ ), Load natural frequency ( $W$ ), and load velocity ( $C$ ), are also taken into consideration. The solutions are obtained by using the Mindlin–Goodman method, Galerkin method, Laplace integral transformation, and theory of convolution. Maple software was used to calculate response amplitudes and presented graphically. The plots clearly show that as the damping coefficient increases, the corresponding deflection of the beam subjected to harmonic moving loading diminishes substantially. It was also noted that the deflection profiles decrease as the foundation stiffness and shear modulus increase. It was also discovered that as the axial force, load natural frequency, and velocity of the moving load increase, the deflection profiles decrease. The findings also revealed that the load natural frequency and damping coefficient have more significant effects on the structure subjected to a harmonically varying moving load compared with other structural parameters. Hence, to reduce the adverse effects of resonance and guarantee the safety of lives and properties, higher values of those parameters are recommended for the designers of such structures.*

### \*Author for Correspondence

Ajoge E.O.  
E-mail: [cajoge@oauife.edu.ng](mailto:cajoge@oauife.edu.ng)

<sup>1</sup>Research Professor Department of Mathematics and Statistics, Conference University of Science and Technology, Osara, Kogi State, Nigeria

<sup>2</sup>Research Scholar, Department of Centre for Energy Research and Development, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria

<sup>3</sup>Research Scholar, Department of Biochemistry, Chemistry, and Physics, College of Science and Mathematics, Georgia Southern University, 1332 Southern Drive Statesboro, GA 30458, USA

<sup>4</sup>Research Scholar, Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria.

<sup>5</sup>Research Scholar, Department of Mechanical and Materials Engineering, University of Turku, Finland

<sup>6</sup>Research Scholar, Department of Material Science and Engineering, Obafemi Awolowo University, Ile-Ife, Nigeria

Received Date: October 30, 2025

Accepted Date: October 31, 2025

Published Date: December 12, 2025

**Citation:** Jimoh A., Ajoge E.O., Ajiola D.I., Nicholas O.O., Adeyemi S.S., Abdulkabir L.A. Flexural Vibration of Non-Uniform Beam on Bi-Parametric Foundation Under Harmonic Moving Load with Non-Classical Boundary Conditions. International Journal of Mechanical Dynamics and Systems Analysis. 2025; 3(2): 28-42p

**Keywords:** Non-uniform beam, time-dependent boundary conditions, harmonic moving load, bi-parametric foundation, damping coefficient

## INTRODUCTION

The movement of individuals and materials has long been recognized as a core element of societal functioning. Consequently, rising population levels necessitate the development of advanced machines and vehicles to meet evolving technological demands. The study of moving loads on structural

---

members has attracted the attention of many researchers [1–10] in the fields of Engineering, Physics, and Mathematical Physics. This emphasis arises from the need to safeguard human life and physical assets, which remains of paramount concern.

The researchers considered beams whose properties do not vary with position  $x$  along the span  $L$  of the beam. In real-life situations, these properties vary along the span  $L$  of the beam, which makes the partial differential equation describing the dynamical system difficult to solve because the equation will then have both singular and variable coefficients.

To address these challenges, Jimoh and Ajoge [11] employed the Galerkin method to investigate the dynamic behavior of a non-uniform Rayleigh beam supported by a Pasternak foundation and subjected to a harmonically varying moving load. Their results showed that increasing the axial force, foundation stiffness, damping coefficient, shear modulus, and rotary inertia reduced the deflection profile of the beam. They also noted that the influence of rotary inertia and damping was more pronounced than that of other structural parameters. In a related study, Awodola et al. [12] examined the vibration of a non-uniform Bernoulli–Euler beam on a Pasternak foundation under moving distributed masses and loads of varying magnitude. They used Galerkin with a series representation of the Heaviside function, Laplace transformation, and finite elements with Newmark methods to obtain closed-form solutions and presented the results graphically. Their findings revealed that increases in the axial force, shear modulus, and foundation modulus led to decreases in the response amplitude of the beam. Adedowole and Famuagun [13] examined the behavior of simply supported non-uniform prismatic beams placed on variable elastic foundations and exposed to harmonic loads in motion. They used the generalized Galerkin method (GGM) to obtain a solution. They established the resonance phenomenon and conditions under which it occurred. Oni and Awodola [14] used the GGM and the Strubles asymptotic technique to assess the vibration of a non-uniform Rayleigh beam resting on a variable elastic foundation subjected to a moving load. Oni and Omolofe [15] analyzed the dynamic behavior of non-uniform Bernoulli–Euler beams subjected to concentrated loads travelling at variable speeds. They employed Galerkin and integral transformation techniques to treat the second-order partial differential equation describing the dynamic system. Their findings revealed that the critical velocity of the system increased as the values of the foundation stiffness increased, which reduced the risk of resonance. Hsu [16] employed a spline collocation method to investigate the vibrational response of a non-uniform beam resting on an elastic foundation. Some of the other researchers who considered a non-uniform beam in their works are Olotu et al. [17], Oni and Jimoh [18], Omolofe et al. [19], Robinson and Adali [20], and Jimoh et al. [21].

It is worth noting that all the researchers considered classical boundary conditions, particularly simply supported boundary conditions, and under such conditions, both the displacement and the bending moment vanish at both ends of the dynamical system. Research works in which other classical boundary conditions, such as clamped-clamped boundary conditions, clamped-free boundary conditions, and sliding boundary conditions in which the deflection and slope, bending moment and shear force, and slope, and shear force vanish, respectively, are not common in the literature. Some researchers who considered boundary conditions in their studies include Awodola et al. [12], Ozdemir and Kava [22], Lee [23], Somia [24], and Mirzaei et al. [25], to name a few.

Another interesting type of boundary condition in the study of dynamical systems is non-classical boundary conditions, which are time-dependent and elastically supported boundary conditions. In real-life situations, the boundary conditions are not homogeneous, and vibration problems frequently deal with continuous systems in which one or more boundaries are not stationary but constrained to undergo a sort of displacement that varies with time. This is the case of time-dependent boundary conditions, and practical examples of such dynamic systems are long bridges partly or wholly erected in the earthquake region and traversed by vehicles or equipment in large industries. Because of the complexity and difficulty involved in the formulation and solution of the governing partial differential equations describing such dynamical systems, research on non-classical boundary conditions is scarce in

literature. Some of the few researchers who considered time-dependent boundary conditions in their studies are Mindlin and Goodman [26], Ajibola [27], and Adedowole [28]. Research involving time-dependent boundary conditions and harmonic moving loads remains outstanding in literature.

Thus, in this study, we examined the dynamic response to harmonically varying moving loads of non-uniform elastic beams resting on a bi-parametric foundation with non-classical boundary conditions and time-dependent boundary conditions. In addition, the damping term is incorporated into the governing equation that describes the dynamic problem.

### The Initial Boundary-Valued Problem

The governing partial differential equation for a non-uniform beam of length  $L$  on a bi-parametric foundation traversed by a moving load  $P(x, t)$  of mass  $m$  moving with velocity  $c$ , and the damping term included is given by Frybal [29]:

$$\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 V(x, t)}{\partial x^2} \right) + \mu(x) \frac{\partial^2 V(x, t)}{\partial t^2} - N \frac{\partial^2 V(x, t)}{\partial x^2} + \varepsilon \frac{\partial V(x, t)}{\partial t} - G \frac{\partial^2 V(x, t)}{\partial x^2} + KV(x, t) = P(x, t) \quad (1)$$

Where,

$x$  is the spatial coordinate;  $t$  is the time

$EI(x)$  is the variable flexural rigidity of the structure,  $\mu(x)$  is the variable mass per unit length of the beam,  $I(x)$  is the variable moment of inertia.

$K$  is the foundation stiffness;  $G$  is the shear modulus;  $N$  is the axial force.  
 $\varepsilon$  is the damping coefficient;  $c$  is the velocity of the moving load.  
 $V(x, t)$  is the transverse displacement;  $w$  is the load natural frequency  
 $E$  is the Young's modulus,

The variable mass per unit length  $\mu(x)$  of the beam and the variable moment of inertia are given as

$$\mu(x) = \mu_o \left( 1 + \text{Sin} \frac{\pi x}{L} \right) \quad (2)$$

and

$$I(x) = I_o \left( 1 + \text{Sin} \frac{\pi x}{L} \right)^3 \quad (3)$$

Where,  $\mu_o$  and  $I_o$  are the constant mass per unit length and the constant moment of inertia of the beam, respectively.

Simplifying Equation (3) fully to obtain

$$I(x) = \frac{I_o}{4} \left( 10 + 15 \text{Sin} \frac{\pi x}{L} - \text{Sin} \frac{3\pi x}{L} - 6 \text{Cos} \frac{2\pi x}{L} \right) \quad (4)$$

The moving Load  $P(x, t)$  is assumed to be harmonically, which is of the form

$$P(x, t) = P \cos wt \delta(x - ct) \quad (5)$$

Where,  $P$  is the magnitude of the load, and  $\delta(x - ct)$  is the Dirac delta function defined as

$$\delta(x - ct) = \begin{cases} 0, & x \neq ct \\ 1, & x = ct \end{cases} \quad (6)$$

With properties

$$\text{i. } \delta(-x) = \delta(x) \quad (7)$$

$$\text{ii. } \int_a^b \delta(x - ct) f(x) dx = f(ct) \quad (8)$$

By substituting Equations (2), (4), and Equations (5) into (1), we obtain

$$\begin{aligned} & \frac{EI_0}{4} \frac{\partial^2}{\partial x^2} \left[ \left( 10 + 15 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6 \cos \frac{2\pi x}{L} \right) \frac{\partial^2 V(x,t)}{\partial x^2} \right] \\ & + \mu_0 \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 V(x,t)}{\partial t^2} + \varepsilon \frac{\partial V(x,t)}{\partial t} - N \frac{\partial^2 V(x,t)}{\partial x^2} - G \frac{\partial^2 V(x,t)}{\partial x^2} \\ & + KV(x,t) = P \cos \omega t \delta(x-ct) \end{aligned} \quad (9)$$

After simplification and rearrangement, Equation (9) can be written as

$$\begin{aligned} & \frac{EI_0}{4} \left[ \frac{\pi^2}{L^2} \left( -15 \sin \frac{\pi x}{L} + 9 \sin \frac{3\pi x}{L} + 24 \cos \frac{2\pi x}{L} \right) \frac{\partial^2 V(x,t)}{\partial x^2} \right] \\ & + \frac{2\pi}{L^2} \left( 15 \cos \frac{\pi x}{L} - 3 \cos \frac{3\pi x}{L} + 12 \sin \frac{2\pi x}{L} \right) \frac{\partial^3 V(x,t)}{\partial x^3} \\ & + \left( 10 + 15 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6 \cos \frac{2\pi x}{L} \right) \frac{\partial^4 V(x,t)}{\partial x^4} \\ & + \mu_0 \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 V(x,t)}{\partial t^2} + \varepsilon \frac{\partial V(x,t)}{\partial t} - (N + G) \frac{\partial^2 V(x,t)}{\partial x^2} \\ & + KV(x,t) = P \cos \omega t \delta(x-ct) \end{aligned} \quad (10)$$

The boundary conditions corresponding to Equation (10) were time-dependent. That is, at each point of the boundary points, there are two boundary conditions, which can be written as

$$D_i[V(0,t)] = f_i(t), i = 1, 2 \quad (11)$$

$$D_i[V(L,t)] = f_i(t), i = 1, 2 \quad (12)$$

Where,  $D_i$  are linear homogeneous differential operators of order less than or equal to three.

The initial conditions of motion at time  $t = 0$  are specified by two arbitrary functions given as

$$V(x,0) = V_0(0) \quad (13)$$

$$\frac{\partial V(x,t)}{\partial t} = V_0(0) \quad (14)$$

### Operational Simplification of the Governing Equation

To solve the nature of the problem in Equation (10) above, the analytical solution to the non-homogeneous initial boundary value problem Equation (10) with non-homogeneous boundary conditions Equation (11) and Equation (12) and non-homogeneous initial conditions Equation (13) and Equation (14) are sought.

To this end, an approach based on Mindlin–Goodman was extended, and we proceeded as follows:

An auxiliary variable  $U(x,t)$  in the form

$$V(x,0) = U(x,0) + \sum_{i=0}^4 f_i(t) g_i(x) \quad (15)$$

is introduced.

Substituting Equation (15) into Equation (10) transforms the boundary-valued problem into terms of  $V(x,t)$  to the boundary-valued problem in terms of  $U(x,t)$ . The functions  $g_i(x)$  are called the displacement influence function and are to be chosen to render the boundary conditions for the boundary-valued problem in  $U(x,t)$  homogeneous, while the function  $f_i(x)$  is called the pertinent displacement at the respective boundaries.

Substituting Equation (15) into Equation (10) to obtain

$$\frac{EI_0}{4\mu_0} \left( 10 + 15 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6 \cos \frac{2\pi x}{L} \right) \frac{\partial^4 U(x,t)}{\partial x^4}$$

$$\begin{aligned}
 & + \frac{EI_0}{2L\mu_0} \left( 15 \cos \frac{\pi x}{L} - 3 \cos \frac{3\pi x}{L} + 12 \sin \frac{2\pi x}{L} \right) \frac{\partial^3 U(x,t)}{\partial x^3} \\
 & + \left[ \frac{EI_0 \pi^2}{4\mu_0 L^2} \left( -15 \sin \frac{\pi x}{L} + 9 \sin \frac{3\pi x}{L} + 24 \cos \frac{2\pi x}{L} \right) - (N + G) \right] \frac{\partial^2 U(x,t)}{\partial x^2} \\
 & + \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 U(x,t)}{\partial t^2} + \frac{\varepsilon}{\mu_0} \frac{\partial U(x,t)}{\partial t} + \frac{K}{\mu_0} U(x,t) \\
 & = P \cos \omega t \delta(x - ct) \\
 \\
 & - \sum_{i=1}^4 \left[ \begin{aligned}
 & \frac{EI_0}{4\mu_0} \left( 10 + 15 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6 \cos \frac{2\pi x}{L} \right) \frac{\partial^4 f_i(t) g_i(x)}{\partial x^4} \\
 & + \frac{EI_0}{2L\mu_0} \left( 15 \cos \frac{\pi x}{L} - 3 \cos \frac{3\pi x}{L} + 12 \sin \frac{2\pi x}{L} \right) \frac{\partial^3 f_i(t) g_i(x)}{\partial x^3} \\
 & + \left( \frac{EI_0 \pi^2}{4\mu_0 L^2} \left( -15 \sin \frac{\pi x}{L} + 9 \sin \frac{3\pi x}{L} + 24 \cos \frac{2\pi x}{L} \right) - (N + G) \right) \frac{\partial^2 f_i(t) g_i(x)}{\partial x^2} \\
 & + \left( 1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 f_i(t) g_i(x)}{\partial t^2} + \frac{\varepsilon}{\mu_0} \frac{\partial f_i(t) g_i(x)}{\partial t} + \frac{K}{\mu_0} f_i(t) g_i(x)
 \end{aligned} \right] \quad (16)
 \end{aligned}$$

The expression in Equation (15) must satisfy the boundary conditions Equations (11) and (12). Consequently, we have

$$D_i[U(0, t)] + \sum_{j=0}^4 f_j(t) D_i[g_j(0)] = f_i(t), \quad i = 1, 2 \quad (17)$$

$$D_i[U(L, t)] + \sum_{j=0}^4 f_j(t) D_i[g_j(L)] = f_i(t), \quad i = 3, 4 \quad (18)$$

Substituting Equation (15) into the initial conditions, Equations (13) and (14), we obtain

$$U(x, 0) = - \sum_{i=0}^4 f_i(0) g_i(x) \quad (19)$$

$$\frac{\partial U(x,0)}{\partial t} = - \sum_{i=0}^4 \dot{f}_i(0) g_i(x) \quad (20)$$

Using the Mindlin–Goodman method [26], the boundary conditions Equations (11) and (12) in terms of  $U(x, t)$  can be made homogeneous if the function  $g_i(x)$  is chosen such that the conditions given by:

$$D_i[g_j(0)] = \delta_{ij} \quad (i = 1, 2 \quad j = 1, 2, 3, 4) \quad (21)$$

and

$$D_i[g_j(L)] = \delta_{ij} \quad (i = 1, 2 \quad j = 1, 2, 3, 4) \quad (22)$$

are satisfied.

Where,

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad (23)$$

Equations (21) and (22) are used in the non-homogeneous boundary conditions, Equations (17) and (18), to obtain the homogeneous boundary conditions given as

$$D_i[U(0, t)] = 0, \quad i = 1, 2 \quad (24)$$

$$D_i[U(L, t)] = 0, \quad i = 1, 2 \quad (25)$$

The problem is reduced to solving the non-homogeneous partial differential Equation (16) subject to the homogeneous boundary conditions Equations (24) and (25) with the non-homogeneous initial conditions Equations (19) and (20).

### Solution Procedure

To solve the initial boundary-valued problem Equation (16), we employed Galerkin's method by assuming a solution of the form.

$$U(x, t) = \sum_{m=1}^n Y_m(t) U_m(x) \quad m = 1, 2, \dots \quad (26)$$

Where,  $U_m(x)$  is chosen as a suitable kernel of Galerkin's method such that the given boundary conditions are satisfied.

$$U_m(x) = \text{Sin} \frac{m\pi x}{L} \quad (27)$$

It is important to note that the required  $g_i(x)$  and  $f_i(t)$  were  $g_1(x)$ ,  $g_3(x)$ ,  $f_1(t)$  and  $f_3(t)$ .

Thus, we can write

$$f_1(t) = \text{Sin}\Omega t \text{ and } f_3(t) = e^{-\beta t} \text{Sin}\Omega t \quad (28)$$

Where,  $\Omega$  and  $\beta$  are frequency and parameter, respectively.

Differentiating Equation (28) with respect to  $t$  to obtain

$$\left. \begin{aligned} \dot{f}_1(t) &= \Omega \text{Cos}\Omega t, \ddot{f}_1(t) = -\Omega^2 \text{Sin}\Omega t A = \pi r^2 \\ \dot{f}_3(t) &= -\beta e^{-\beta t} \text{Sin}\Omega t + \Omega e^{-\beta t} \text{Cos}\Omega t \\ \ddot{f}_3(t) &= \beta^2 e^{-\beta t} \text{Sin}\Omega t - 2\beta \Omega e^{-\beta t} \text{Cos}\Omega t - \Omega^2 e^{-\beta t} \text{Sin}\Omega t \end{aligned} \right\} \quad (29)$$

Also,

$$\frac{\partial^4 f_i(t) g_i(x)}{\partial x^4} = f_i(t) \frac{\partial^4 g_i(x)}{\partial x^4} = 0 \quad (30a)$$

$$\frac{\partial^3 f_i(t) g_i(x)}{\partial x^3} = f_i(t) \frac{\partial^3 g_i(x)}{\partial x^3} = 0 \quad (30b)$$

$$\frac{\partial^2 f_i(t) g_i(x)}{\partial x^2} = f_i(t) \frac{\partial^2 g_i(x)}{\partial x^2} = 0 \quad (30c)$$

$$\frac{\partial^2 f_i(t) g_i(x)}{\partial t^2} = g_i(x) \frac{\partial^2 f_i(t)}{\partial t^2} = -2\beta \Omega e^{-\beta t} \text{Cos}\Omega t \quad (30d)$$

$$\frac{\partial f_i(t) g_i(x)}{\partial t} = g_i(x) \frac{\partial f_i(t)}{\partial t} = \Omega \text{Cos}\Omega t + (\Omega \text{Cos}\Omega t - \beta \text{Sin}\Omega t) e^{-\beta t} \quad (30e)$$

$$f_i(t) g_i(x) = (1 + e^{-\beta t}) \text{Sin}\Omega t \quad (30f)$$

Substituting Equations (30a)–(30f) into Equation (16) to obtain

$$\begin{aligned} & \frac{EI_0}{4\mu_0} \left( 10 + 15 \text{Sin} \frac{\pi x}{L} - \text{Sin} \frac{3\pi x}{L} - 6 \text{Cos} \frac{\pi x}{L} \right) \frac{\partial^4 U(x, t)}{\partial x^4} \\ & + \frac{EI_0}{2L\mu_0} \left( 15 \text{Cos} \frac{\pi x}{L} - 3 \text{Cos} \frac{3\pi x}{L} + 12 \text{Sin} \frac{2\pi x}{L} \right) \frac{\partial^3 U(x, t)}{\partial x^3} \\ & + \left( \frac{EI_0}{4\mu_0} \frac{\pi^2}{L^2} \left( -15 \text{Sin} \frac{\pi x}{L} + 9 \text{Sin} \frac{3\pi x}{L} + 24 \text{Cos} \frac{2\pi x}{L} \right) - (N + G) \right) \frac{\partial^2 U(x, t)}{\partial x^2} \\ & + \left( 1 + \text{Sin} \frac{\pi x}{L} \right) \frac{\partial^2 U(x, t)}{\partial t^2} + \frac{\varepsilon}{\mu_0} \frac{\partial U(x, t)}{\partial t} + \frac{K}{\mu_0} U(x, t) \\ & = \frac{P}{\mu_0} \cos \omega t \delta(x - ct) \\ & - \left( 1 + \text{Sin} \frac{\pi x}{L} \right) \left( (-\Omega^2 \text{Sin}\Omega t) + (\beta^2 - \Omega^2) e^{-\beta t} \text{Sin}\Omega t - 2\beta \Omega e^{-\beta t} \text{Cos}\Omega t \right) \\ & - \frac{\varepsilon}{\mu_0} \left( \Omega \text{Cos}\Omega t + (\Omega \text{Cos}\Omega t - \beta \text{Sin}\Omega t) e^{-\beta t} \right) + \frac{K}{\mu_0} (1 + e^{-\beta t}) \text{Sin}\Omega t \end{aligned} \quad (31)$$

Substituting Equation (26) into Equation (31), after simplification and rearrangement, we obtain:

$$\begin{aligned} & \left(1 + \sin \frac{\pi x}{L}\right) \sin \frac{m\pi x}{L} \ddot{Y}_m(t) + A_1 \sin \frac{m\pi x}{L} \dot{Y}_m(t) \\ & + \left[ \begin{array}{l} A_2 \left(10 + 15 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6 \cos \frac{2\pi x}{L}\right) \sin \frac{m\pi x}{L} \\ A_3 \left(15 \cos \frac{\pi x}{L} - 3 \cos \frac{3\pi x}{L} + 12 \sin \frac{2\pi x}{L}\right) \cos \frac{m\pi x}{L} \\ A_4 \left(-15 \sin \frac{\pi x}{L} + 9 \sin \frac{3\pi x}{L} + 24 \cos \frac{2\pi x}{L}\right) \sin \frac{m\pi x}{L} \\ + A_5 \sin \frac{m\pi x}{L} + A_6 \sin \frac{m\pi x}{L} \end{array} \right] Y_m(t) \\ & = \frac{P}{\mu_o} \cos \omega t \delta(x - ct) - B_1(t) \left(1 + \sin \frac{\pi x}{L}\right) + B_2(t) + B_3(t) \end{aligned} \quad (32)$$

Where,

$$\left. \begin{aligned} P &= Mg, A_1 = \frac{\varepsilon}{\mu_o}, A_2 = \frac{EI_o}{4\mu_o} \left(\frac{m\pi}{L}\right)^4, A_3 = \frac{EI_o}{2L\mu_o} \left(\frac{m\pi}{L}\right)^2 \\ A_4 &= \frac{EI_o}{4\mu_o} \left(\frac{\pi}{L}\right)^2 \left(\frac{m\pi}{L}\right)^2, A_5 = (N + G) \left(\frac{m\pi}{L}\right)^2, A_6 = \frac{K}{\mu_o} \\ B_1(t) &= -\Omega^2 \sin \Omega t + (\beta^2 - \Omega^2) e^{-\beta t} \sin \Omega t - 2\beta \Omega e^{-\beta t} \cos \Omega t \\ B_2(t) &= -\frac{\varepsilon}{\mu_o} (\Omega \cos \Omega t + (\Omega \cos \Omega t - \beta \sin \Omega t) e^{-\beta t}) \\ B_3(t) &= \frac{K}{\mu_o} (1 + e^{-\beta t}) \sin \Omega t \end{aligned} \right\} \quad (33)$$

To determine an expression for  $Y_m(t)$ , it is required that the expression on the Right Hand Side (RHS) of (32) is orthogonal to the function  $U_k(x)$ , Hence, we have

$$\int_0^L \sum_m^n \left\{ \begin{array}{l} \left(1 + \sin \frac{\pi x}{L}\right) \sin \frac{m\pi x}{L} \ddot{Y}_m(t) + A_1 \sin \frac{m\pi x}{L} \dot{Y}_m(t) \\ + \left[ \begin{array}{l} A_2 \left(10 + 15 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6 \cos \frac{2\pi x}{L}\right) \sin \frac{m\pi x}{L} \\ A_3 \left(15 \cos \frac{\pi x}{L} - 3 \cos \frac{3\pi x}{L} + 12 \sin \frac{2\pi x}{L}\right) \cos \frac{m\pi x}{L} \\ A_4 \left(-15 \sin \frac{\pi x}{L} + 9 \sin \frac{3\pi x}{L} + 24 \cos \frac{2\pi x}{L}\right) \sin \frac{m\pi x}{L} \\ + A_5 \sin \frac{m\pi x}{L} + A_6 \sin \frac{m\pi x}{L} \end{array} \right] Y_m(t) \end{array} \right\} U_k(x) = 0 \quad (34)$$

The initial conditions are given as

$$Y_m(0) = 0 = \dot{Y}_m(0) \text{ and } U_k(0) = 0 = \dot{U}_k(0) \quad (35)$$

The appropriate function for the  $K^{th}$ . The normal mode for the vibration of the beam is given as:

$$U_k(x) = \sin \lambda_k(x) + A_k \cos \lambda_k(x) + B_k \sinh \lambda_k(x) + C_k \cosh \lambda_k(x) \quad (36)$$

For the simply supported beam

$$A_k = B_k = C_k = 0 \text{ and } \lambda_k = \frac{k\pi}{L} \quad (37)$$

and Equation (36) becomes

$$U_k(x) = \sin \frac{k\pi x}{L} \quad (38)$$

Hence, Equation (34) can now be written as

$$\begin{aligned} \ddot{Y}_m(t) + \alpha_1 \dot{Y}_m(t) + \alpha_2 Y_m(t) &= \frac{P}{D_1 \mu_o} \cos \omega t \sin \theta_c t \\ + A_9 \sin \Omega t + A_{10} e^{-\beta t} \sin \Omega t + A_{11} e^{-\beta t} \cos \Omega t \end{aligned} \quad (39)$$

Where,

$$\left. \begin{aligned} \alpha_1 &= \frac{D_2}{D_1}, \alpha_2 = \frac{D_3}{D_1}, \theta_c = \frac{k\pi c}{L} \\ A_9 &= -\frac{A_7\Omega}{D_1} - \frac{A_8\varepsilon\Omega}{\mu_0 D_1} + \frac{A_7 k}{\Omega D_1} \end{aligned} \right\} \quad (40)$$

$$A_{10} = -\frac{A_7}{D_1}(\beta^2 - \Omega^2) - \frac{A_8\varepsilon\beta}{\mu_0 D_1} + \frac{A_8 k}{\mu_0 D_1} \quad (41a)$$

$$\left. \begin{aligned} A_{11} &= \frac{2A_7\Omega\beta}{D_1}, A_7 = \frac{2L}{k\pi} + \frac{L}{2}, A_8 = \frac{2L}{k\pi} \\ D_1 &= \int_0^L \left(1 + \sin \frac{\pi x}{L}\right) \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \\ D_2 &= A_1 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \end{aligned} \right\} \quad (41b)$$

$$\begin{aligned} D_{3A} &= A_1 \int_0^L A_2 \left(10 + 15\sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} - 6\cos \frac{2\pi x}{L}\right) \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \\ &+ A_4 \int_0^L \left(-15\sin \frac{\pi x}{L} + 9\sin \frac{3\pi x}{L} + 24\cos \frac{2\pi x}{L} + A_5 + A_6\right) \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \end{aligned} \quad (41c)$$

$$D_{3B} = A_3 \int_0^L \left(15\cos \frac{\pi x}{L} - \cos \frac{3\pi x}{L} + 12\sin \frac{2\pi x}{L}\right) \cos \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \quad (41d)$$

$$D_3 = D_{3A} + D_{3B} \quad (41e)$$

Equation (39) is the transformed equation governing the dynamic system.

### Solution of the Transformed Equation

To solve Equation (39), we apply the Laplace transformation, defined as

$$F(s) = \int_0^L e^{-st} f(t) dt \quad (42)$$

Hence, we subject Equations (39) to (42) in conjunction with the initial conditions in Equation (35), after simplification and rearrangement to obtain

$$Y_m(s) = \left(\frac{1}{r_1+r_2}\right) \left[ \frac{R}{2} \left( \frac{w+\theta_c}{s^2+(w+\theta_c)^2} - \frac{w-\theta_c}{s^2+(w-\theta_c)^2} \right) + A_9 \left( \frac{\Omega}{s^2+\Omega^2} \right) \right] \left( \frac{1}{s+r_2} - \frac{1}{s+r_1} \right) + A_{10} \left( \frac{\Omega}{(s+\beta)^2+\Omega^2} \right) + A_{11} \left( \frac{s+\beta}{(s+\beta)^2+\Omega^2} \right) \quad (43)$$

To obtain the Laplace inversion of Equation (43), we use the convolution integral defined as:

$$f(t) * g(t) = \int_0^L f(u)g(t-u)du \quad (44)$$

By applying Equation (44), the Laplace inversion of Equation (43) can be obtained as follows:

$$Y_m(t) = \phi \left[ (I_1 - I_6) + (I_2 - I_7) + (I_3 - I_8) + (I_4 - I_9) + (I_5 - I_{10}) \right] \quad (45)$$

Where,

$$\phi = \frac{1}{r_1+r_2} \quad (46)$$

$$I_1 = -\frac{R}{2} \int_0^t \sin(w + \theta_c)u e^{-r_2(t-u)} du \quad (47a)$$

$$I_2 = -\frac{R}{2} \int_0^t \sin(w + \theta_c)u e^{-r_1(t-u)} du \quad (47b)$$

$$I_3 = A_9 \int_0^t \sin \Omega u e^{-r_2(t-u)} du \quad (47c)$$

$$I_4 = A_{10} \int_0^t \sin \Omega u e^{-(r_2-\beta)u-r_2t} du \quad (47d)$$

$$I_5 = A_{11} \int_0^t \cos \Omega u e^{-(r_2-\beta)u-r_2t} du \quad (47e)$$

$$I_6 = \frac{R}{2} \int_0^t \sin(w + \theta_c)u e^{-r_1(t-u)} du \quad (47f)$$

$$I_7 = -\frac{R}{2} \int_0^t \sin(w - \theta_c)u e^{-r_1(t-u)} du \quad (47g)$$

$$I_8 = A_9 \int_0^t \text{Sin } \Omega u e^{-r_1(t-u)} du \quad (47h)$$

$$I_9 = A_{10} \int_0^t \text{Sin } \Omega u e^{-(r_1-\beta)u-r_1t} du \quad (47i)$$

$$I_{10} = A_{10} \int_0^t \text{Cos } \Omega u e^{-(r_1-\beta)u-r_1t} du \quad (47j)$$

By evaluating the integrals Equations (47a)–(47j) and substituting the results into Equation (45), we obtain:

$$Y_m(t) = \phi \left\{ \begin{aligned} & \frac{R}{2} \left[ \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_2t} - \text{Cos}(w-\theta_c)t + \frac{r_2 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right. \\ & \left. - \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_2t} - \text{Cos}(w-\theta_c)t + \frac{r_2 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right] \\ & - \frac{R}{2} \left[ \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_1t} - \text{Cos}(w-\theta_c)t + \frac{r_1 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right. \\ & \left. - \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_1t} \text{Cos}(w-\theta_c)t + \frac{r_1 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right] \\ & + A_9 \left[ \frac{\Omega}{\Omega+r_2^2} \left( e^{-r_2t} - \text{Cos } \Omega t + \frac{r_2 \text{Sin } \Omega t}{\Omega} \right) \right. \\ & \left. - \frac{\Omega}{\Omega+r_1^2} \left( e^{-r_1t} - \text{Cos } \Omega t + \frac{r_1 \text{Sin } \Omega t}{\Omega} \right) \right] \\ & + A_{10} \left[ \frac{1}{\Omega+r_2-\beta} \left( e^{-r_2t} - e^{-\beta t} \text{Cos } \Omega t + \frac{(r_2-\beta)e^{-\beta t} \text{Sin } \Omega t}{\Omega} \right) \right. \\ & \left. - \frac{1}{\Omega+r_1-\beta} \left( e^{-r_1t} - e^{-\beta t} \text{Cos } \Omega t + \frac{(r_1-\beta)e^{-\beta t} \text{Sin } \Omega t}{\Omega} \right) \right] \\ & + A_{11} \left[ \frac{1}{\Omega+r_2-\beta} \left( e^{-\beta t} \text{Sin } \Omega t + \frac{(r_2-\beta)e^{-\beta t} \text{Cos } \Omega t}{\Omega} \right) \right. \\ & \left. - \frac{1}{\Omega+r_1-\beta} \left( e^{-\beta t} \text{Sin } \Omega t + \frac{(r_1-\beta)e^{-\beta t} \text{Cos } \Omega t}{\Omega} \right) \right] \end{aligned} \right\} \quad (48)$$

Substituting Equations (27) and (48) into Equation (26) to obtain

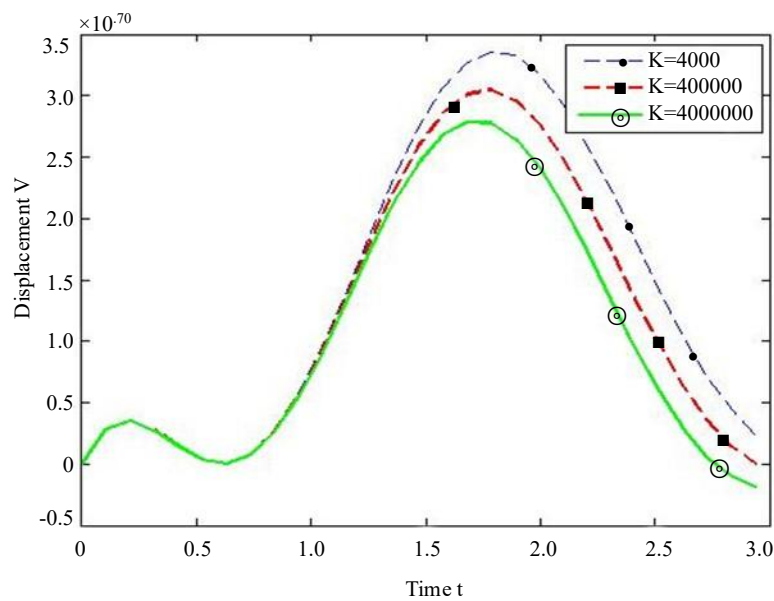
$$U(x, t) = \sum_{m=1}^n \phi \left\{ \begin{aligned} & \frac{R}{2} \left[ \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_2t} - \text{Cos}(w-\theta_c)t + \frac{r_2 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right. \\ & \left. - \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_2t} - \text{Cos}(w-\theta_c)t + \frac{r_2 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right] \\ & - \frac{R}{2} \left[ \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_1t} - \text{Cos}(w-\theta_c)t + \frac{r_1 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right. \\ & \left. - \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2} \left( e^{-r_1t} \text{Cos}(w-\theta_c)t + \frac{r_1 \text{Sin}(w-\theta_c)t}{w-\theta_c} \right) \right] \\ & + A_9 \left[ \frac{\Omega}{\Omega+r_2^2} \left( e^{-r_2t} - \text{Cos } \Omega t + \frac{r_2 \text{Sin } \Omega t}{\Omega} \right) \right. \\ & \left. - \frac{\Omega}{\Omega+r_1^2} \left( e^{-r_1t} - \text{Cos } \Omega t + \frac{r_1 \text{Sin } \Omega t}{\Omega} \right) \right] \\ & + A_{10} \left[ \frac{1}{\Omega+r_2-\beta} \left( e^{-r_2t} - e^{-\beta t} \text{Cos } \Omega t + \frac{(r_2-\beta)e^{-\beta t} \text{Sin } \Omega t}{\Omega} \right) \right. \\ & \left. - \frac{1}{\Omega+r_1-\beta} \left( e^{-r_1t} - e^{-\beta t} \text{Cos } \Omega t + \frac{(r_1-\beta)e^{-\beta t} \text{Sin } \Omega t}{\Omega} \right) \right] \\ & + A_{11} \left[ \frac{1}{\Omega+r_2-\beta} \left( e^{-\beta t} \text{Sin } \Omega t + \frac{(r_2-\beta)e^{-\beta t} \text{Cos } \Omega t}{\Omega} \right) \right. \\ & \left. - \frac{1}{\Omega+r_1-\beta} \left( e^{-\beta t} \text{Sin } \Omega t + \frac{(r_1-\beta)e^{-\beta t} \text{Cos } \Omega t}{\Omega} \right) \right] \end{aligned} \right\} \text{Sin } \frac{m\pi x}{L} \quad (49)$$

Equation (49) represents the transverse displacement of the non-uniform elastic beam resting on a bi-parametric foundation and subjected to a harmonically varying moving load with time-dependent boundary conditions.

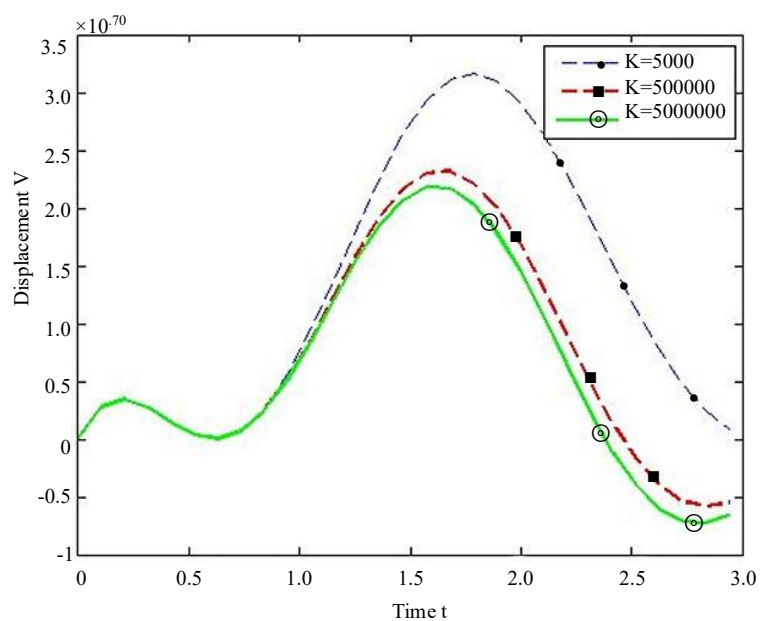
## DISCUSSIONS

In Figure 1, the impact of foundation stiffness on the dynamic response of a non-uniform beam supported by a bi-parametric foundation and subjected to a harmonic moving load with time-dependent boundaries is presented. The plot shows a continuous reduction in deflection corresponding to higher foundation stiffness values.

Figure 2 depicts the effect of shear modulus on the beam, which shows that as the values of the shear modulus significantly increase, there is no noticeable effect at the beginning, sharp decreases in the middle, and then slight decreases in the deflection profiles of the beam.



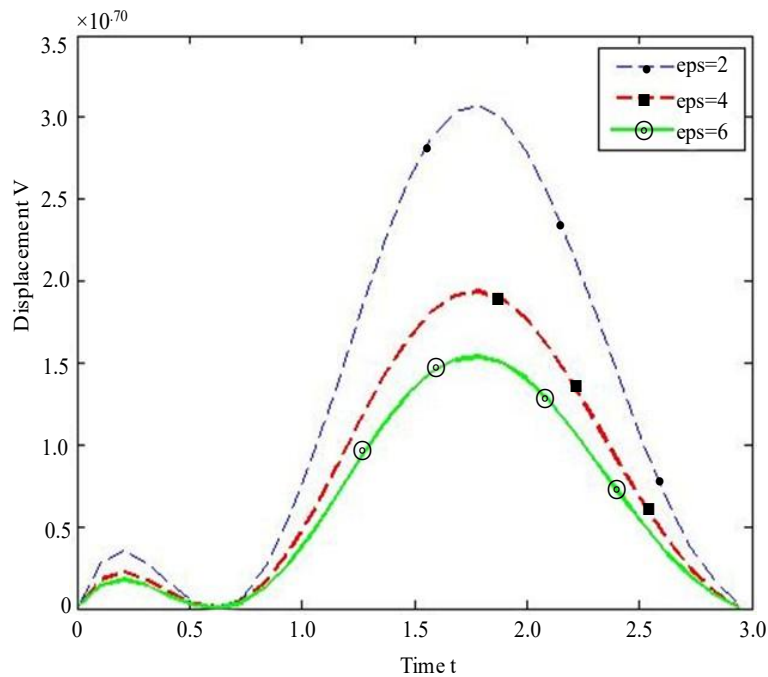
**Figure 1.** Deflection profile of a non-uniform elastic beam resting on bi-parametric foundations subjected to a harmonically varying moving load with time-dependent boundary conditions for various values of foundation stiffness ( $K$ ).



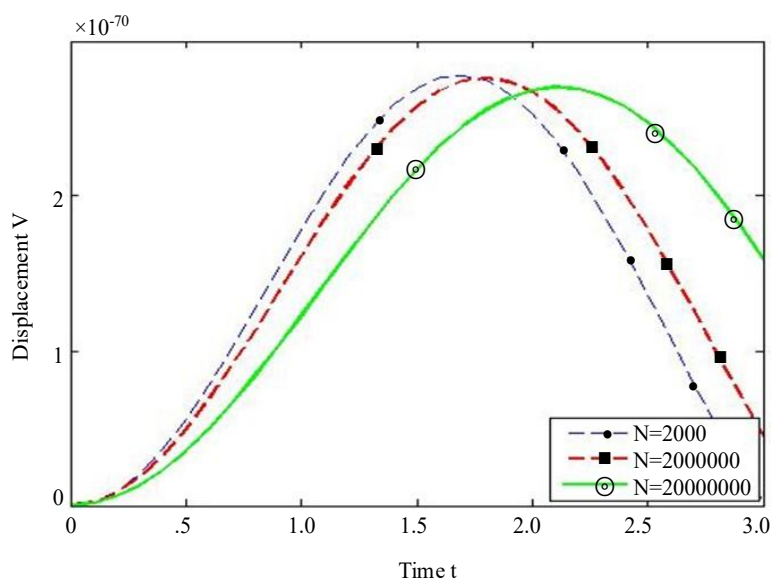
**Figure 2.** Deflection profile of a non-uniform elastic beam resting on bi-parametric foundations subjected to a harmonically varying moving load with time-dependent boundary conditions for various values of shear modulus ( $G$ ).

The effect of the damping coefficient on the dynamical system is shown in Figure 3, where it can be seen from the figure that as the values of the damping coefficient slightly increase, there is a slight noticeable effect at the beginning regarding the deflection, and then sharp decreases in the deflection profile in the middle.

Figure 4 shows the effect of axial force on the beam subjected to a moving load. As the values of the axial force increase, there is initially a slight decrease in the response amplitude of the beam and a sharp decrease in the deflection profile of the beam, especially towards the middle.

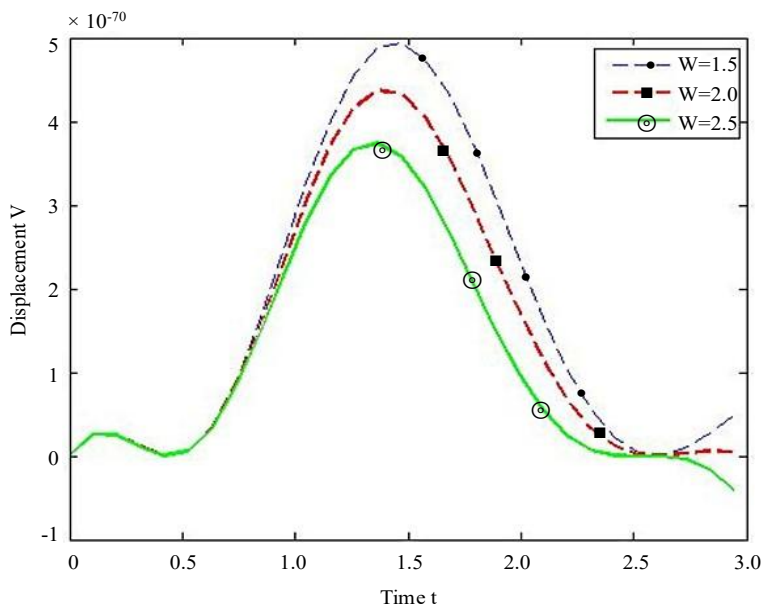


**Figure 3.** Deflection profile of a non-uniform elastic beam resting on bi-parametric foundations subjected to a harmonically varying moving load with time-dependent boundary conditions for varying damping coefficient ( $\alpha$ ).

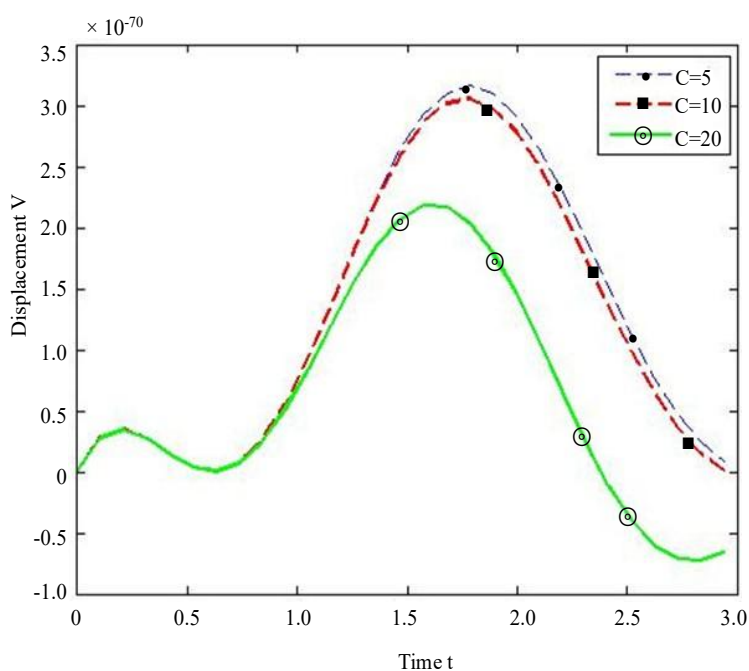


**Figure 4.** Deflection profile of a non-uniform elastic beam resting on bi-parametric foundations subjected to a harmonically varying moving load with time-dependent boundary conditions for various values of axial force ( $N$ ).

Figures 5 and 6 show the effects of the load natural frequency and load velocity, respectively, on the beam subjected to a moving load. As shown in Figure 5, as the values of the load natural frequency increase slightly, the response amplitude of the beam steadily decreases, and this is not noticeable at the beginning but becomes noticeable in the middle. As shown in Figure 6, as the load velocity significantly increased, the deflection profile slightly decreased. The effects were not observed at the beginning, but there was a steady decrease in the deflection in the middle, followed by a sharp decrease in the response amplitude of the beam.



**Figure 5.** Deflection profile of a non-uniform elastic beam resting on bi-parametric foundations subjected to a harmonically varying moving load with time-dependent boundary conditions for various values of load natural frequency (W).



**Figure 6.** Deflection profile of a non-uniform elastic beam resting on two-parameter foundations subjected to a harmonically varying moving load with time-dependent boundary conditions for various values of load velocity (c).

## CONCLUSION

The dynamic analysis of a non-uniform elastic beam with time-dependent boundary conditions when the beam is under the action of a harmonically varying moving load is considered in this study. The system is described by a non-homogeneous fourth-order partial differential equation whose coefficients are both variable and singular, and the associated boundary conditions are non-homogeneous. The main objective is to obtain an analytical solution to the dynamical problem because such solutions shed more light on vital information regarding the flexural vibration of the system. Beams with properties such as moment of inertia and mass per unit length varying with position  $x$  along span  $L$  were considered, and the beam was assumed to be reinforced and resting on a bi-parametric foundation. The damping effect was also incorporated into the governing equation that describes the system.

The solution techniques are based on the Mindlin–Goodman method, Galerkin’s method, integral transformation, and the theorem of convolution. Mindlin–Goodman was extended to transform the governing non-homogeneous partial differential equation with non-homogeneous boundary conditions to a non-homogeneous partial differential equation with homogeneous boundary conditions. Galerkin’s method was applied to eliminate the singular behavior present in the governing equation and convert it into a set of second-order differential equations. Following this reduction, integral transform techniques and the convolution theorem are used to derive the analytical form of the system response. The resulting transverse displacement of the beam was evaluated and illustrated using graphical plots, from which the following observations were obtained:

- a. An increase in the damping coefficient leads to a corresponding decrease in the displacement response of the non-uniform beam subjected to harmonic moving loads under time-dependent boundary conditions.
- b. When a beam with time-dependent boundary conditions rests on a bi-parametric foundation, higher values of foundation stiffness, shear modulus, and axial force consistently produce lower response amplitudes.
- c. For fixed values of the other structural parameters, the transverse displacements of the beam decrease as the load natural frequency and load velocity increase.

Finally, it is observed from the findings that the load natural frequency and damping coefficient have more noticeable and significant effects on the dynamical problem compared with the effects of other structural parameters on the system. Hence, designers of such structures should take note of these parameters during the implementation stage. This will go a long way to reduce the resonance effect and guarantee the safety of life.

## REFERENCES

1. Oyelade AO, Sadiq OM. Enhanced damping characteristics of Timoshenko beam on elastic and metal material foundation. *J Serbian Soc Comput Mech.* 2020;14(1):52–62. doi:10.24874/jsscm.2020.14.01.05.
2. Ajijola OO. Dynamic response to moving load of prestressed damped shear beam resting on bi-parametric elastic foundation. *Afr J Math Stat Stud.* 2024;7(4):328–342. doi:10.52589/AJMSS-0JZLWMLB.
3. Jimoh A, Ajoge E. Effects of two-parameter foundation on uniform elastic beam subjected to concentrated moving loads with clamped–clamped boundary conditions. *Int J Mech Des.* 2025;11:14–26.
4. Previati G, Ballo F, Stabile P. Beams on elastic foundation: a variable reduction approach for nonlinear contact problems. *Eur J Mech A Solids.* 2025;111:105514. doi:10.1016/j.euromechsol.2024.105514.
5. Siqueira LO, Cortez RL, Hoefel SS. Vibration analysis of an axial-loaded Euler–Bernoulli beam on two-parameter foundation. In: *Proceedings of the 25th ABCM International Congress of Mechanical Engineering (COBEM 2019); 2019 Oct 20–25; Uberlândia, MG, Brazil.* 2019.

6. Yahya A, Zakria A, Ahmed IE, Ahmed SA, Dargail HE, Hassan AA, Salih E. Viscoelastic Pasternak foundation analysis of a thermoelastic microbeam using Moore–Gibson–Thompson heat conduction under Klein–Gordon nonlocality. *AIMS Math.* 2025;10:10340–10358. doi:10.3934/math.2025471.
7. Arathy M, Bennet K. Analysis of beam on elastic foundation. *Int J Creat Res Thoughts.* 2024;12(1):20–30.
8. Abbas W, Bakr OK, Nassar MM, Abdeen MAM, Shabrawy M. Analysis of tapered Timoshenko and Euler–Bernoulli beams on an elastic foundation with moving loads. *J Math.* 2021;2021:1–13. doi:10.1155/2021/6616707.
9. Mahmut TO, Veysel K, Mustafa OY, Mardani A. Vibration analysis of steel fibre reinforced self-compacting concrete beam on elastic foundation. *Comput Concr.* 2021;27(2):85–97.
10. Ogunbamike OK. Seismic analysis of simply supported damped Rayleigh beams on elastic foundation. *Asian Res J Math.* 2020;16(11):1–11.
11. Jimoh A, Ajoge EO. Dynamic response of non-uniform Rayleigh beam subjected to harmonically varying moving load. *J Appl Math Comput.* 2018;2(8):345–356. doi:10.26855/jamc.2018.08.004.
12. Awodola TO, Awe BB, Jimoh SA. Vibration of non-uniform Bernoulli–Euler beam under moving distributed masses resting on Pasternak elastic foundation subjected to variable magnitude. *Afr J Math Stat Stud.* 2024;7(1):1–19. doi:10.52589/AJMSS-ZQCKBA87.
13. Adedowole A, Famuagun KS. Simply supported non-uniform prismatic beam resting on variable elastic foundation carrying traveling harmonic loads. *J Emerg Trends Eng Appl Sci.* 2017.
14. Oni ST, Awodola TO. Vibration under moving load of a non-uniform Rayleigh beam on variable elastic foundation. *J Niger Assoc Math Phys.* 2003;7:191–206.
15. Oni S, Omolofe B. Dynamic behaviour of non-uniform Bernoulli–Euler beams subjected to concentrated loads travelling at varying velocities. *J Niger Assoc Math Phys.* 2008;9. doi:10.4314/jonamp.v9i1.40037.
16. Hsu MH. Vibration analysis of non-uniform beam resting on elastic foundation using the spline collocation method. *Tamkang J Sci Eng.* 2009;12(2):113–122.
17. Olotu OT, Agboola OO, Gbadeyan JA. Free vibration analysis of non-uniform Rayleigh beams on variable Winkler elastic foundation using differential transform method. *Ilorin J Sci.* 2021;8(1):1–20. doi:10.54908/iljs.2021.08.01.001.
18. Oni ST, Jimoh A. Dynamic response to moving concentrated loads of non-uniform simply supported prestressed Bernoulli–Euler beam resting on bi-parametric subgrades. *Int J Sci Eng Res.* 2016;7(3):754–770.
19. Omolofe B, Oni ST, Tolorunshaga JM. On the transverse motions of non-uniform prismatic deep beam under the actions of variable magnitude moving loads. *Lat Am J Solids Struct.* 2009;6:153–167.
20. Robinson MTA, Adali S. Buckling of nonuniform and axially functionally graded nonlocal Timoshenko nanobeams on Winkler–Pasternak foundation. *Compos Struct.* 2018;206:95–103. doi:10.1016/j.compstruct.2018.07.046.
21. Jimoh SA, Ogunbamike OK, Olanipekun A. Dynamic response of non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity. *Asian Res J Math.* 2018;9(4):1–18. doi:10.9734/ARJOM/2018/41327.
22. Özdemir Ö, Kaya MO. Flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli–Euler beam by differential transform method. *J Sound Vib.* 2006;289(1–2):413–420. doi:10.1016/j.jsv.2005.01.055.
23. Lee HP. Dynamic response of a beam with a moving mass. *J Sound Vib.* 1996;191:289–294. doi:10.1006/jsvi.1996.0122.
24. Somia A. The rope wear analysis with the use of 3D vision system. *Control Eng.* 2013;60(5):48–49.
25. Mirzaei MMH, Loghman A, Arefi M. Effect of temperature dependency on thermoelastic behavior of rotating variable thickness FGM cantilever beam. *J Solid Mech.* 2019;11(3):657–669.

26. Mindlin RD, Goodman LE. Beam vibrations with time-dependent boundary conditions. *J Appl Mech.* 1950;17:377–380. doi:10.1115/1.4010161.
27. Ajibola SO. Dynamic analysis under moving concentrated loads of Rayleigh beams with time dependent boundary conditions [PhD thesis]. Akure (Nigeria): Federal University of Technology Akure; 2009.
28. Adedowole A. Flexural motions under moving distributed masses of beam-type structures on Vlasov foundation and having time dependent boundary conditions [PhD thesis]. Akure (Nigeria): Federal University of Technology Akure; 2016.
29. Frýba L. *Vibration of solids and structures under moving loads.* Groningen: Springer Netherlands; 1972. doi:10.1007/978-94-011-9685-7.