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COMPARATIVE EVALUATION OF FIXED-POINT THEOREMS INVESTIGATING DIFFERENCES AMONG CONDITIONS IN METRIC SPACES

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Abstract

This paper provides a concise overview of metric spaces, their fundamental properties, and the idea of continuity as it pertains to these spaces, drawing parallels to uniform continuity and convergence. Also included is a review of previous research on metric space, its generalization, and practical outcomes from studies that utilized these spaces in various applications. We introduce a fixed point theorem for mixed monotone mappings in partially ordered metric spaces, assuming weak contractility. This theorem extends current knowledge and has implications for *various mathematical and applied problems. Our findings include new results that enhance the theoretical framework. Under the condition of weak contractility, we also establish a fixed point theorem for mixed monotone mappings in metric spaces with partial order. In addition to incorporating several new findings, our theory can be applied to a wide range of problem classes. We address the uniqueness of solution for a periodic boundary value problem as an application and talk about the existence of such a solution.*

Keywords: *Comparative Evaluation, Fixed-Point Theorems, Metric Spaces, Logical Programming,Topology*

1. INTRODUCTION

Mathematicians have given this structure a lot of thought because the fixed point hypothesis in standard metric space has advanced. The idea of metric space is a powerful tool for addressing specific issues in advanced mathematics. Since the idea of metric spaces has been the subject of much study. After that, only a small number of papers have successfully implemented fixed point hypothesis in those kinds of spaces. Consequently, spaces play a key role in both topology and certain techniques and issues based on logical programming. Metric spaces were initially investigated in 1905 by the eminent French mathematician Maurice Frechet; the term "metric" refers to distance on a number line.In this case, the metric space is defined by Azam et al., which demonstrates that the results of a fixed point on a mapping also follow the rational and defined inequality conditions for that space Abdeljawad T.(2013) [1]. After that, theory on cone metric space is generated and initialized. Following that, additional publications have examined the theory's fixed point generation without verifying it with a complicated highly regarded metric space.

Space scholars have been working diligently lately attempting to furnish the semantics area with an idea of distance. In particular, in his exploration on the denotational semantics of dataflow networks, Matthews laid out the idea of a fractional metric space and found, in addition to other things, a lovely correspondence between these spaces and what are known as weightable quasimetric spaces. Altun I and *Durmaz G* (2009) [2] Program check can profit from his exhibit of a speculation of the Banach constriction planning theorem to the fractional metric setting. Fixed point theorems in halfway metric spaces were consequently demonstrated by various scholars.*Ciepliński K. (2012)*[3]

$$d(Tx, Ty) \leq \emptyset(m(x, y))$$

Normal fixed point consequences of mappings in metric spaces have been acquired utilizing contractive circumstances with the purported correlation capability ϕ of the structure since the renowned work of Boyd and Wong. *Kirk WA. (2004)* [4] Capability ϕ was accepted diversely in various articles. These suspicions weren't generally expressed unequivocally, and infrequently they were major areas of strength for excessively. This incorporates a choice of current fixed point results inside fractional metric spaces.

Reasoning normal fixed point results in halfway metric spaces and officially expressing various potential circumstances for an examination capability are the objectives of this exploration. These outcomes are similarly relevant to standard metric spaces, since they are excellent circumstances.*O'Regan D and Petruşel A.(2008)* [5] We will show through models that there are situations where a halfway metric outcome can be utilized rather than the ordinary metric one.

1.1.Objectives of the Study

- To determine which fixed-point theorems for metric spaces are the most important.
- To study the variations of these assumptions that affect the presence of fixed points, for example, contraction constants, completeness, compactness, and others.

2. LITERATURE REVIEW

Ahmed, M. A. (2011) concoct a fixed point theorem for a summed up withdrawal in disjoined semi metric spaces. Moreover, we characterize a solitary fixed point for each planning and depict it. Thirdly, in completely separated semi metric spaces, we lay out an extra fixed point theorem. Various theorems in[1-3,5,7,8,11-14,16,19,20,22] are summed up by these theorems. Besides, we offer a few comments in regards to [17, Theorem 3] and [21, Theorem 1].*Ahmed MA.(2011)*[6]

Kirk, W. A. (2004) demonstrated a bounded open set in a complete space that is not expanding always has a fixed point if there exists such that for all. We further prove that for any complete tree with and for every closed convex subset that is geodesically bounded, there exists a fixed point for the continuous mapping that maps to it. Also, for continuous mappings, the fixed point property is present in a geodesically bounded complete tree. It is from these latter findings that several versions of graph theory's traditional fixed edge theorem are derived. *Park S. (2002)* [7]

Subrahmanyam, P. V. (2018) Considers a few fixed-point theorems and their applications. The book will function as a subjective sampling of subjects in fixed point theorems, with the themes chosen mostly based on personal tastes.*Singh SL.et al.,(2012)* [8] Neither the most important fixed point theorems nor their most broad formulations are covered in this volume. There is zero effort to set the issues discussed in a chronological or historical context. Still, here's hoping it adds to the existing literature and piques readers' interests in doing more research. *Farmakis I and Moskowitz M(2013)* [9]

Singh, S. L. (2012) proposed a fixed-point theorem for two multivalued maps on a total metric space, expanding on a past consequence of Đorić and Lazović (2011) for a multivalued map on a metric space that fulfills Ćirić-Suzuki-type summed up withdrawal. Furthermore, we determine a speculation of Čirić's (1974) critical normal fixed point theorem as a particular occurrence. A class of practical conditions that emerge in powerful programming *Nachbar J. (2010)* [10] and whether they have a typical arrangement is likewise covered. *Franklin JN.(2002)* [11]

Ciepliński (2012) directed research. In 1991, J.A. Bread cook utilized a variation of Banach's fixed point theorem to decide the steadiness of a useful condition in a solitary variable. This technique is presently the second most well known method for demonstrating the Hyers-Ulam strength of useful conditions.*Liu Z. et al.,(2011)*[12] Then again, most of journalists utilize a theorem by Diaz and Margolis, as Radu did. Introducing utilizations of different fixed point theorems to the hypothesis of the Hyers-Ulam strength of utilitarian conditions is the fundamental motivation behind this review.*Proinov PD (2006)* [13]

3. PRELIMINARIES

Definition 1: In a fractional metric space, X and p are nonempty sets and p is a halfway metric on X, and that truly intends that for any x, y, and z in X, there exists a capability p : X × X → R.

$$
\begin{aligned}
&(p_1)\ x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y),\\
&(p_2)\ p(x,x) \le p(x,y),\\
&(p_3)\ p(x,y) = p(y,x),\\
&(p_4)\ p(x,y) \le p(x,z) + p(z,y) - p(z,z).
\end{aligned}
$$

In light of p1 and p2, it is apparent that x = y if px, y = 0. The factors x and px, nonetheless, probably won't approach zero.

The T0 geography τp on X is comprised of the group of open p-balls {Bpx, ε : x ∈ X, ε > 0}, where Bpx, ε = {y ∈ X : px, y < px, x=ε} for all x ∈ X and ε > 0. It is created for each incomplete metric p on X. On the off chance that limn → ∞px, xn px, x, the succession {xn} in X, where p is a component of X, merges to a point x in X concerning τp. This will be composed as xn → x as n approaches limitlessness or as limn → ∞xn = x. For each arrangement {xn} in X, if T: X → X is consistent at x0 ∈ X in τp, then

$$x_n \longrightarrow x_0 \implies Tx_n \longrightarrow Tx_0.$$

Remark: A uniqueness condition is clearly excessive for a grouping's breaking point in a halfway metric space. As a delineation, suppose that X=(0+∞) and p(x,y) and px, y = max{x, y} for all x and y in X. Then, for each {xn} = {1}, p(xn,x=x=p9x,x) and, thusly, xn → 2 and xn → 3 as n → ∞. What's more, the capability p doesn't be guaranteed to must be constant as in pxn, yn → px, y and xn → x and yn → y.

Definition 2: Let X, p be a fractional metric space, as characterized in Definition 2. Then, we have this.

Expecting limn,m → ∞pxn, xm exists and is limited, a grouping {xn} in X, p is alluded to as a Cauchy succession.

2 Expecting that p(x, x) =limn,m → ∞pxn, xm, then, at that point, any Cauchy grouping {xn} in X joins to a point x ∈ X with respect to τp, we say that the space X, p is finished.

Each total fractional metric space has an undeniable full subset. The accompanying capability ps : X × X → R+ is characterized when p is a fractional metric on X.

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

One measure on X is Abstract and Applied Analysis. In addition, the limit limn → ∞psxn, where x = 0 as long as *Subrahmanyam PV. Et al., (2013)* [14]

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

4. AUXILIARY THEOREM

Functions ∅:[0,+∞)→[0,+∞] will be examined for the features listed below."ϕn" will represent the nth time iteration of "∅^n ϕ":

(I) $\varphi(t) < t$ for each $t > 0$ and $\varphi^n(t) \to 0, n \to \infty$ for each $t \geq 0$,

(II) φ is nondecreasing and $\varphi^n(t) \to 0, n \to \infty$ for each $t \geq 0$,

(III) φ is right-continuous, and $\varphi(t) < t$ for each $t > 0$,

(IV) φ is nondecreasing and $\sum_{n\geq 1} \varphi^n(t) < +\infty$ for each $t \geq 0$.

Definition 3: 1. Part II is equivalent to Part I.

2. Nondecreasing ϕ + (III) ⇒ (II).

3. (IV) ⇒ Equation II.

4. Even if ϕ is not decreasing, (III) and (IV) cannot be compared.

Proof:

1. Assume that for all t0 ∈ 0 ∞, there exists an II such that ϕt0 ≥ t0. This means that ϕ2t0 ≥ ϕt0 ≥ t0 if ϕ is monotonic. It is impossible for ∅^n (t_0)→0,n→∞ since continuing by induction we obtain that ∅^2 (t_0)≥ ∅(t_0)≥ t_0 and so on.
2. 2 Suppose that III is true and that ϕ is not decreasing. For every fixed t > 0, the nonincreasing and nonnegative sequence {ϕn(t)} is defined by the monotonicity of ϕ, which means that there exists a limit 〖lim〗 _(n→∞) ∅^n (t)= α≥U. Let us assume that α is greater than zero. Next, we can deduce from III that

$$0 < \alpha \leq \lim_{n\to\infty} \varphi^{n+1}(t) = \lim_{\varphi^n(t)\to\alpha+} \varphi(\varphi^n(t)) = \varphi\left(\lim_{\varphi^n(t)\to\alpha+} \varphi^n(t)\right) = \varphi(\alpha), \qquad (3.1)$$

This contradicts the statement that ϕt is less than t.

3. No brainer.
4. Here is an example to illustrate it.

Illustration: This is where

$$\varphi(t) = \begin{cases} \frac{1}{3}t, & 0 \leq t < 1, \\ \frac{1}{2}, & t = 1, \\ \frac{2}{3}t, & t > 1, \end{cases}$$

meets prerequisites IV however not III. The nondecreasing capability Ø(t)=t/(1+t) meets condition III however not IV since Ø^n (t)=t/(1+nt) doesn't exist. The accompanying lemma and comparative affirmations were used and demonstrated in many articles to make different fixed statement results.

Explanation: Characterize X as a metric space of aspect d and consider {yn} as a succession in X fulfilling the condition that {d(y_(n+1),y_n)} is nonincreasing.

$$\lim_{n \to \infty} d(y_{n+1}, y_n) = 0.$$

If the series {y2n} is not Cauchy, then there is a positive integer ε and two positive integer subsequences {mk} and {nk} with the end goal that the accompanying four arrangements approach ε = 0 as k methodologies vastness:

$$d(y_{2m_k}, y_{2n_k}), \qquad d(y_{2m_k}, y_{2n_k+1}), \qquad d(y_{2m_k-1}, y_{2n_k}), \qquad d(y_{2m_k-1}, y_{2n_k+1}).$$

5. THE SCHAUDER-TYCHONOFF FIXED POINT THEOREM

Definition 4: In a Banach space X and a Banach space Y, where C is a subspace of X, a guide f: C → Y is viewed as smaller if and provided that it decreases limited sets to reduced ones somewhat. On the off chance that f is an individual from L(X, Y), it is comparable to saying that the picture of the unit shut ball under f is moderately smaller. As far as groupings, f is conservative if and provided that the succession f(xn) has a merged aftereffect for each limited grouping xn. *Van An T. et al.,(2013)* [15]

Theorem [Schaefer]: Let X be a Banach space and f: X → X be nonstop and minimized. Expect further that the set has limits. When this is valid, f has a fixed point.

$$F = \{x \in X \, : \, x = \lambda f(x) \quad \textit{for some} \quad \lambda \in [0,1]\}$$

Remark: If we are able to demonstrate a priori estimations on the set of all potential fixed points of λf, then the theorem above will be true. Partial differential equations are a common use of this method, which entails estimating a solution to an equation and then using those estimates to establish that the solution exists.

proof Let the map be defined as r> 〖sup〗 _(x∈F) ||x||.

$$g(x) = \begin{cases} f(x) & \text{if} \quad \|f(x)\| \leq 2r \\ \dfrac{2rf(x)}{\|f(x)\|} & \text{if} \quad \|f(x)\| > 2r. \end{cases}$$

Consequently, the function g: BX(0, 2r) → BX(0, 2r) is compact and continuous. The existence of an element x0 ∈ BX(0, 2r) such that g(x0) = x0 is proven by Theorem 1.18. If kf(x0)k is less than or equal to 2r, then

$$x_0 = \lambda_0 f(x_0) \qquad \text{with} \qquad \lambda_0 = \frac{2r}{\|f(x_0)\|} < 1$$

Despite x0 being a member of F, this fact is used to force ||x0|| = 2r. Thus, we obtain that g(x0) = f(x0) = x0.

6. CONCLUSION

In this study, fixed-point theorems in metric spaces are compared and evaluated. Using uniform continuity as a comparison, it explains the fundamentals of metric spaces, convergence, and continuity. Previous work on metric spaces, as well as generalizations of these spaces and practical applications of these conclusions, are also reviewed in the study. On a metric space with halfway request, the creators show a fixed-point theorem for a blended droning planning under the frail contractility condition. Among many other things, the theorem can be used to find and prove that periodic boundary value problems have unique solutions. Finding the best fixed-point theorems for metric spaces and looking at how different circumstances affect the presence of fixed points are the main goals of this research.

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