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Expanding Hicks Contraction Theory, Multi-Valued Mappings in Generalized b-Menger Spaces

Pradip Kumar Keer[1,*], Geeta Agrawal[2]

Abstract

This paper extends the Hicks contraction theory to multi-valued mappings within generalized b-Menger spaces. By introducing new definitions and conditions, we provide a comprehensive analysis of how these contractions behave in broader contexts. We present novel theorems and proofs that generalize existing results, offering insights into the stability and convergence of multi-valued mappings. Examples illustrate the applicability of our theoretical advancements, demonstrating their relevance to diverse metric frameworks. This work opens new avenues for research in contraction theory and metric space analysis.

Keywords: Hicks contractions, multi-valued mappings, generalized b-Menger spaces.

INTRODUCTION

Contraction mappings are central to fixed point theory and metric space analysis, with Hick's contractions playing a pivotal role in understanding fixed points under various conditions [1]. These mappings have been widely studied due to their robustness and significant applications across diverse mathematical problems [2]. Recent research has expanded the theory to encompass multi-valued mappings, where a single point in the domain can correspond to multiple points in the codomain. This extension introduces new complexities and broadens the scope of fixed-point theorems [3, 4]. Multi-valued Hick's contractions, in particular, are being explored for their potential to solve problems where traditional single-valued assumptions are inadequate [5, 6]. The study of Hicks contractions in b-Menger spaces, a generalization of metric spaces, has also progressed b-Menger spaces offer a more flexible framework by relaxing the standard metric conditions, allowing for broader types of contractions and applications [7, 8]. Recent work has delved into these generalized metrics and their implications for fixed point theory [9, 10].

This paper aims to extend Hick's contraction theory to multi-valued mappings within generalized b-Menger spaces. We propose new definitions and extend existing results, providing a comprehensive understanding of contractions in these broader frameworks. Our findings reflect the latest advancements and offer new perspectives for theoretical exploration and practical applications [11–15].

***Author for Correspondence**
Pradip Kumar Keer
E-mail: pradip.keer@gmail.com

[1]Research Scholar, Department of Mathematics, Government Motilal Vigyan Mahavidyalaya (Govt. MVM), Bhopal, Madhya Pradesh, India
[2]Professor, Department of Mathematics, Government Motilal Vigyan Mahavidyalaya (Govt. MVM), Bhopal, Madhya Pradesh, India

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PRELIMINARIES

We will now introduce key notations, definitions, and topological properties of b-Menger spaces. For further details, refer to relevant literature [16-20].

Definition 1 [21]

A mapping $Y: [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm (shortly t-norm) if for each $u, v, w \in [0,1]$ the following conditions are satisfied

(a) $Y(u,v) = Y(v,u)$;

(b) $Y(u), Y(v,w) = Y(Y(u,v),w)$;

(c) $Y(u,v) \leq Y(u,w)$ for $v \leq w$;

(d) $Y(u,1) = Y(1,u) = u$.

Among the most used t-norm: $Y_M(u,v) = \min(u,v)$ and $Y_L(u,v) = \max(u+v-1,0)$.

Definition 2 [21]

A t-norm $\daleth$ is said to be H-type if the family $\{ \daleth^n(x)\}_{n\in N}$ is equi-continuous at the point $x = 1$, which mean that

$\forall \in (0,1), \exists\, \lambda \in (0,1): t > 1 - \lambda \Longrightarrow \daleth^n(t) > 1 - \varepsilon$ for all $n \geq 1$.

Definition 3 [21]

A t-norm $\daleth$ is said to be k-convergent if for all $k \in (0,1)$ we have

$\operatorname{Lim}_{n\to\infty} {}_{i=n}^{\infty}\daleth\left(1 - k^i\right) = 1.$

We should note if $\daleth$ is k- convergent then,

$\forall \delta \in (0,1), \exists v \in \mathbb{N}: \daleth_{i=1}^{n}\left(1 - k^{v+i}\right) > 1 - \delta, \forall\, n \in \mathbb{N}.$

As well, if the t-norm $\daleth$ is k-convergent then $sup_{0\leq x<1}\daleth(x,x) = 1.$

Definition 4 [21]

A quadratic $(\varepsilon, F, \daleth, s)$ is called a, b-Menger space if ε is an arbitrary set, F is a mapping from $\varepsilon \times \varepsilon$ into Δ^+, $\daleth$ is a t-norm and $s \geq 1$ is a real number, such that for all $a, b, c \in \varepsilon$ and $u, v > 0$. We have

(a) $F_{a,a} = \varepsilon_0$;

(b) $F_{a,b} \neq \varepsilon_0\ if\ a \neq b,$

(c) $F_{a,b} = F_{b,a}$;

(d) $F_{a,b}\big(s(u+v)\big) \geq \daleth\Big(F_{a,c}(u), F_{c,b}(v)\Big).$

It is obvious that a Menger space is also a, b-Menger space with the constant $s = 1$. M barki et al.in [5] proved that if $(\varepsilon, F, \daleth, s)$ is a b-Menger space with a continuous t-norm $\daleth$, then $(\varepsilon, F, \daleth, s)$ is a Hausdorff topological space in the topology induced by the family of (ε, λ) −neighborhoods

$\mathcal{N} = \{N_p(\varepsilon, \lambda): p \in \varepsilon, \varepsilon > 0$ and $\lambda > 0\},$

Where $\mathcal{N}_p\{q \in \varepsilon: F_{p,q}(\varepsilon) > 1 - \lambda.$

Definition 5 [21]

Let $(\varepsilon, F, \daleth, s)$ be a, b-Menger space with $\daleth$ is a continuous t-norm, a sequence $\{u_n\} in\ \varepsilon$ is

1. Convergent to $u \in \varepsilon$ if for any given $\varepsilon > 0\ and\ \lambda > 0$ there exist $n_{\varepsilon,\lambda} \in \mathbb{N}$ such that $F_{u_{n,u}}(\lambda) > 1 - \varepsilon$, whenever $n \geq n_{\varepsilon,\lambda}$.
2. A Cauchy sequence if for any $\varepsilon > 0 and\ \lambda > 0$ then there exist $n_{\varepsilon,\lambda} \in \mathbb{N}$ such that $F_{u_{n,u_m}}(\lambda) > 1 - \varepsilon$, whenever $n, m \geq n_{\varepsilon,\lambda}$.
   A b-Menger space $(\varepsilon, F, \daleth, s)$ is complete if each Cauchy sequence in ε is convergent to some point ε.

MAIN RESULT

In this paper, we establish a new fixed-point theorem for multivalued (ν, C)-contractions in b-Menger spaces. Before presenting the main result, we provide the following definition. Additionally, we extend the concept of multivalued Hick's contractions.

Definition 1

Let (E, z, Y, s) be a, b-Menger space. A mapping $f: E \rightarrow 2^E$ (the set of non-empty subsets of E) is said to be a multi-valued (v, C) −contraction if there exist a function $v: E \times E \rightarrow [0,1]$ and a constant $C \geq 0$ such that for all $x, y \in E$, the following conditions holds:

$$H_z(f(x), f(y)) \leq v(z(x,y)) + Cz(x,y),$$

Where H_z denotes the Harsdorf metric induced by the b-Menger space. The function v governs the contraction behaviour of the multi-valued map, ensuring that the distance between any two points in the image sets is controlled by the function v and a constant tern C.

Theorem 2 Generalized Theorem

Let (E, z, Y, s) be a complete b-Menger space, and let $f: E \rightarrow E$ be a multi-valued (v, C) −contraction with values in the space of non-empty subsets of E.Suppose the series $\sum_{n=1}^{\infty} s_n v_n(u)$ converges for some $u > 1$ with $v \in \daleth$. Furthermore, assume that $\lim_{n\to\infty} \sum_{i=1}^{\infty}(1 - v_{n+i-1}(u)) = 1$. Then f admits a fixed point.

Proof:

We prove this theorem by the following steps.

Step 1: Construct a Sequence $\{p_n\}$*,Choose an Initial Point* $p_0 \in E$ *and Select* $p_1 \in f(p_0)$.

Given that $u > 1$, we have $z_{p_{0,p_1}}(u) > 1 - u$. By using the contractively, there exist $p_2 \in f(p_1)$ such that

$$z_{p_{1,p_2}}(v(u)) > 1 - v(u).$$

By Mathematical Induction we can construct a sequence $\{p_n\}$ such that $p_{n+1} \in f(p_n)$ and

$$z_{p_{n,p_{n+1}}}(v_n(u)) > 1 - v(u).$$

For all $n \in N$.

Step 2: Now We Show that $\{p_n\}$ *is a Cauchy Sequence*

Let $\in > 0$ & $\lambda > 0$. Since $\lim_{n\to\infty} \sum_{i=1}^{\infty}(1 - v_{n+i-1}(u)) = 1$,

Then there exist $n_1 \in N$ such that $\lim_{n\to\infty} \sum_{i=1}^{\infty}(1 - v_{n+i-1}(u)) > 1 - \varepsilon$

For all $n \geq n_1$.

On the other hand, side, since $\sum_{n=1}^{\infty} s_n v_n(u)$ converges, there exist $n_2 \in N$ such that

$$\sum_{n=n_2}^{\infty} s_n v_n(u) < \lambda.$$

Let $j = \max(n_1, n_2)$. For each $n \geq j$ and $l \in \mathbb{N}$,we have

$$z_{p_{n,p_{n+1}}}(\lambda) \geq z_{p_{n,p_{n+1}}} \sum_{i=n}^{n+l-1} s_i v_i(u).$$

Now applying the b-Menger triangle inequality then we get

$$z_{p_{n,p_{n+1}}}(\lambda) \geq Y_l\, z_{p_{n,p_{n+1}}} \sum_{i=n}^{n} s_i v_i(u)\, z_{p_{n+1,p_{n+2}}} \sum_{i=n+1}^{n+1} s_i v_i(u), \dots\dots\dots, z_{p_{n+l-1,p_{n+1}}} \sum_{i=n+l-1}^{n+l-1} s_i v_i(u)$$

Since $\sum_{i=n+l-1}^{n+l-1}(1 - v_i(u)$ is closed to $1 - \varepsilon$, we obtain

$$z_{p_{n,p_{n+1}}}(\lambda) \geq Y_l\,(1 - v_n(u), 1 - v_{n+1}(u), 1 - v_{n+l-1}(u) \geq 1 - \varepsilon\,.$$

Therefore $\{p_n\}$ is a Cauchy sequence in E.

Step 3: In This Step We Conclude the Existence of a Fixed Point

Since E is complete $\{p_n\}$ converges to some h and $h \in E$. Now to show that $h \in f(h)$,use the fact that f is a multi-valued mapping and f is closed. for every $\lambda > 0$ and $\varepsilon > 0$, there exist $\theta \in (0,1)$ such that $Y(1-\theta, 1-\theta) > 1-\lambda$.

Given $\varepsilon > 0$, there exist $n_r \in \mathbb{N}$,

$z_{p_{n,}}h(\varepsilon) > 1-\theta.$

Since $p_{n+1} \in f(\,p_n)$ and using the (v, C)-contraction, there exist $y \in f(\,h)$ such that for all $n \geq n_r$,

$z_{p_{n,y}}(\varepsilon) > z_{p_{n,y}}\big(v(\varepsilon)\big) \geq 1 - v(\varepsilon).$

Since $p_n \to h$, it follows that

$\lim_{n\to\infty} z_{p_{n+1,h}}(\varepsilon) \geq 1-\varepsilon.$

Thus $h \in f(\,h)$,
Proving that h is a fixed point of.
Therefore, the theorem is proved

Example 3

Consider the m-b-Menger space (E, z, Y, s), where $E = [0,1], z(x,y) = |x-y|, Y(x,y) = \frac{x+y}{2}$ and $s_n = 1/2^n$. Define the multi-valued mapping $f: E \to 2^E$ as follows:

$f(x) = \{\left[\frac{x}{2}, \frac{x+1}{2}\right]\ if\ x \in [0,1],$

We want to verify that f is a (v, C)-contraction and admits a fixed point.

Let $v(x) = \frac{x}{2}\ and\ c = 0$. For any $x, y \in E$, observe that

$H_z\big(f(x), f(y)\big) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| = v(z(x,y))$

Thus, f is a (v, C)-contraction.

Now choose an initial point $p_0 = 0$, and construct the sequence $\{p_n\}$ according to the theorem:

$p_1 \in f(p_0) = [0,0.5], p_1 = 0.25$

$p_2 \in f(p_1) = [0.125,0.625], p_2 = 0.3125,$

And so on. the sequence $\{p_n\}$ converges to $p = 0.5$, which is the fixed point of f,since

$f(0.5) = [0.25,0.75], and\ 0.5 \in f(0.5).$

Thus, the multi-valued mapping f has a fixed point at $p = 0.5$.

Corollary 4

Let (E, z, Y, s) be a complete b-Menger space, and let g $f: E \to 2^E$ be a multi-valued (v, C)-contraction. I f there exist $u > 1$ such that the series $\sum_{n=1}^{\infty} s_n v_n(u)$ converges and the limit $\lim_{n\to\infty} \sum_{i=1}^{\infty}(1 - v_{n+i-1}(u)) = 1$. Then f admits a fixed point.

This follows that directly from theorem 3.2. By constructing the sequence $\{p_n\}$ as in the proof of the theorem and showing that it is a Cauchy sequence, the completeness of the b-Menger space ensures the existence of a limit point h , which must satisfy $h \in f(h)$, establishing that h is a fixed point of f

Thus, the corollary is proved.

CONCLUSION

In conclusion, this paper successfully extends Hick's contraction theory to multi-valued mappings within generalized b-Menger spaces. By introducing new definitions and conditions, we demonstrated the existence of fixed points under these broader contexts. Our results not only generalize existing theories but also provide deeper insights into the behaviour of multi-valued mappings. This work contributes to the ongoing research in contraction theory and metric spaces, opening avenues for further exploration. The examples provided support the practical relevance of our theoretical findings.

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