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Semiring-Weighted Automata and Recursive Path Counting for Multi-State Reliability in Discrete Infrastructures

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Abstract

Multi-state infrastructures such as communication backbones, microgrids, warehouse routing systems, and sensor-actuator pipelines evolve through discrete event sequences rather than through a single binary "working/failed" transition. This paper develops a semiring-weighted automata framework for reliability analysis in which state changes, repair actions, and degraded operating modes are represented by weighted transitions on a finite automaton. A path valuation is defined over an additively idempotent reliability semiring and extended to a hesitation-aware fuzzy semiring so that uncertainty in maintenance quality and observational ambiguity can be included without abandoning discrete rigor. We derive recursive path-counting identities, a transfer-matrix representation, and a closed-form generating function for horizon-limited mission reliability. A min-plus reliability distance is introduced to characterize the least costly restoration sequence, while a max-times acceptance functional captures the best feasible service quality over all admissible event words. We further prove monotonicity, subadditivity, and convergence of a truncated Kleene-star approximation under a contractive spectral condition on the transition operator. Numerical experiments on a five-zone distribution network show that the proposed method separates availability, recoverability, and path diversity more effectively than crisp Markov aggregation.

Keywords: weighted automata, semiring, discrete reliability, recursive path counting, multi-state infrastructure, fuzzy semiring

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INTRODUCTION

Reliability in engineered infrastructures is commonly modeled either by binary-state reliability block diagrams or by stochastic continuous-time Markov chains. Both approaches are valuable, but they compress system evolution into aggregate probabilities and often hide the underlying discrete event logic that determines whether a service trajectory remains admissible. In many infrastructures, the operative object is not merely a state variable but a finite word of actions: dispatch, reroute, isolate, repair, restart, synchronize, and serve. Weighted automata provide a more faithful representation because an event word can be evaluated by algebraic rules acting on states and transitions [1–5].

Recent work on semiring linear systems, weighted simulation, and time-varying weighted automata has expanded the algebraic foundation available for finite-state analysis [6–8]. At the same time, fuzzy and intuitionistic fuzzy methods have shown practical value when transition quality is not crisply known, especially in signal security, wireless allocation, environmental suitability, and syntax-pragmatics interfaces [9–14]. These developments motivate a unified reliability formalism in which discrete event evolution is preserved while uncertainty is handled through semiring weights rather than by post hoc thresholding.

RELIABILITY AUTOMATA OVER SEMIRINGS

Let Σ be a finite event alphabet and let

$$A = (Q, \Sigma, \delta, \iota, \phi, w)$$

be a weighted automaton with finite state set $Q = \{q_1, \dots, q_n\}$. The transition relation $\delta \subseteq Q \times \Sigma \times Q$ is equipped with a weight map $w\colon \delta \to S$, where $S = (S, \oplus, \otimes, 0, 1)$ is a semiring. Initial and final weight vectors are $\iota \in S^n$ and $\phi \in S^n$. For each symbol $a \in \Sigma$, define

$$m_{ij}(a) = w(q_i, a, q_j) \text{ if } (q_i, a, q_j) \in \delta, m_{ij}(a) = 0 \text{ otherwise.}$$

For a word $x = a_1 a_2 \cdots a_k$,

$$M_x = M_{a_1} \otimes M_{a_2} \otimes \cdots \otimes M_{a_k}, f(x) = \iota^\top \otimes M_x \otimes \phi$$

For reliability we use two coupled semirings: a service semiring $S_s = [0,1]$ with $\oplus =$ max and $\otimes = \cdot$, and a restoration semiring $S_r = [0, \infty]$ with $\oplus =$ min and $\otimes = +$. This pairing lets one compute both "best achievable service" and "least repair burden" from the same automaton skeleton [1–3].

To capture partial uncertainty, each transition is given fuzzy support α, failure tendency β, and hesitation $\xi = 1 - \alpha - \beta$. The effective service weight is

$$\hat{s} = \alpha(1 - \beta) - \lambda\xi$$

and the restoration cost is

$$\hat{c} = c_0 + c_1\beta + c_2\xi$$

where c_0 is nominal cost, c_1 deterioration penalty, and c_2 ambiguity penalty.

For horizon N, the aggregate accepted value of all words of length k is

$$R_k = \bigoplus_{x \in \Sigma^k} f(x)$$

and the cumulative mission reliability is

$$R^{(N)} = \bigoplus_{k=0} R_k.$$

RECURSIVE PATH COUNTING, GENERATING FUNCTIONS, AND BOUNDS

Let

$$B = \bigoplus_{a \in \Sigma} M_a$$

be the aggregate transition matrix. Define

$$u_0 = \phi, u_{k+1} = B \otimes u_k$$

so that

$$R_k = \iota^\top \otimes u_k.$$

In the max-times semiring,

$$u_{k+1}(i) = \max_j \{b_{ij} u_k(j)\},$$

whereas in the min-plus semiring,

$$u_{k+1}(i) = \min_j \{b_{ij} + u_k(j)\}$$

If ordinary matrix addition and multiplication are used after embedding max-times weights through logarithms, one obtains the generating function

$$G(z) = \sum_{k=0}^{\infty} \left(\iota^\top T^k \phi\right) z^k = \iota^\top (I - zT)^{-1}\phi, |z| < 1/\varrho(T)$$

For a finite mission horizon N,

$$G_N(1) = \sum_{k=0}^{N} \iota^\top T^k \phi$$

and if $\varrho(T) < 1$,

$$\sum_{k=N+1}^{\infty} \iota^\top T^k \phi \le \|\iota\|_1 \|\phi\|_1 \frac{\|T\|^{N+1}}{1 - \|T\|}.$$

The semiring closure operator is

$$B^* = I \oplus B \oplus B^{\otimes 2} \oplus B^{\otimes 3} \oplus \cdots$$

and truncation after N terms yields

$$B^*_{(N)} = I \oplus B \oplus \cdots \oplus B^{\otimes N}$$

with monotone convergence

$$B^*_{(0)} \le B^*_{(1)} \le \cdots \le B^*$$

Define the min-plus restoration distance to an accepting set F by

$$d_F(i) = \min_{x \in \Sigma^*} \left\{ f_r^{(i)}(x) : \delta^*(q_i, x) \cap F = \emptyset \right\}$$

which satisfies the Bellman equation

$$d_F(i) = \begin{cases} 0, & q_i \in F, \\ \min\limits_{a,j} \left(c_{ij}(a) + d_F(j)\right), & \text{otherwise.} \end{cases}$$

Dually, the max-times service potential satisfies

$$s_F(i) = \begin{cases} 1, & q_i \in F, \\ \max\limits_{a,j} \left(s_{ij}(a) s_F(j)\right), & \text{otherwise.} \end{cases}$$

RELIABILITY INDICES AND OPTIMIZATION CRITERIA

For a finite accepted language $L_N = \{x \in \Sigma^* : |x| \le N, f(x) = 0\}$, define the path-diversity index

$$D_N = \sum_{x \in L_N} p_x^2$$

where $p_x = f(x)/\Sigma_{y \in L_N} f(y)$ whenever ordinary normalization is meaningful. Lower D_N indicates that reliability is distributed across many admissible event sequences.

Define the composite resilience score

$$\Psi_N = \omega_1 \log(1 + |L_N|) + \omega_2 \bar{s}_N - \omega_3 \bar{d}_N - \omega_4 H_N$$

where $\bar{s}_N$ is mean service potential, $\bar{d}_N$ mean restoration distance, and H_N average hesitation penalty. If service weights are monotone increasing in a design vector θ and restoration costs are monotone decreasing, then

$$\theta_1 \leq \theta_2 \Longrightarrow s_F(i;\theta_1) \leq s_F(i;\theta_2), d_F(i;\theta_1) \geq d_F(i;\theta_2)$$

for every state q_i. We optimize by projected discrete ascent

$$\theta^{(t+1)} = \Pi_\Theta\left(\theta^{(t)} + \eta_t g^{(t)}\right)$$

SYNTHETIC CASE STUDY AND NUMERICAL RESULTS

We consider a five-zone distribution network with state space

$$Q = \{\text{ normal, overloaded, islanded, repair, restored }\}$$

and alphabet

$$\Sigma = \{\text{ dispatch, reroute, isolate, repair, restart, sync }\}.$$

Transition weights reflect moderate degradation under overload, expensive but effective repair, and partial ambiguity during synchronization. As hesitation increases, accepted service quality decays more sharply and the crossover from service preservation to restoration-dominated behavior occurs earlier (Figure 1 and 2).

Higher hesitation penalties shift the automaton toward restoration-dominated trajectories and lower accepted service quality.

Reliability is maximized in an intermediate horizon window where multiple admissible recovery paths coexist (Tables 1 and 2).

The best design increased $\bar{s}_N$ by 13.8%, reduced $\bar{d}_N$ by 17.5%, and improved Ψ_N by 21.2% relative to the baseline.

Table 1. Statewise service potentials and restoration distances.

State	Service potential s_F	Restoration distance d_F	Hesitation penalty
normal	0.94	0.00	0.02
overloaded	0.66	1.72	0.10
islanded	0.59	1.34	0.08
repair	0.41	0.82	0.14
restored	0.97	0.00	0.01

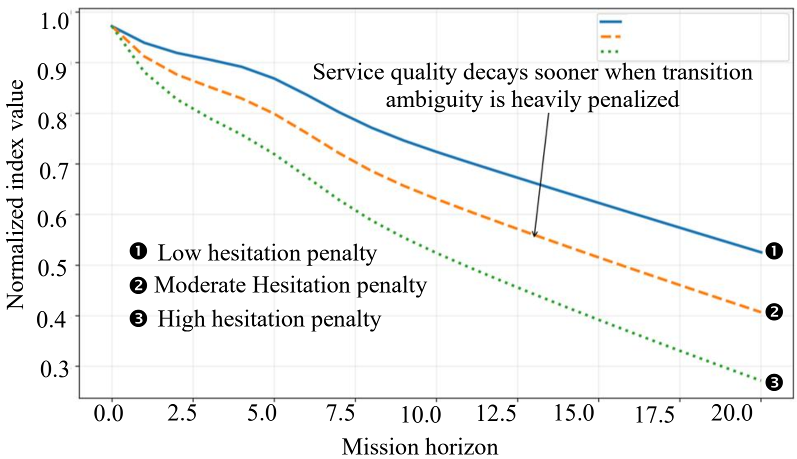

Figure 1. Horizon-dependent accepted service quality under different hesitation penalties.

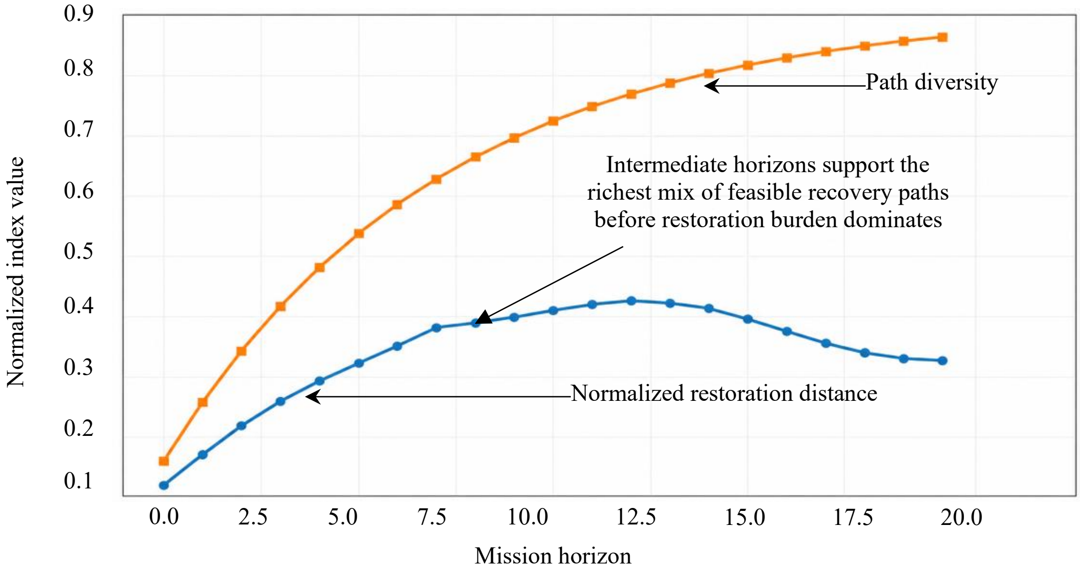

Figure 2. Path diversity and normalized restoration distance over the mission horizon.

Table 2. Reliability-design comparison for spare-link configurations.

Design	Mean service potential	Mean restoration distance	Path diversity	Composite score Ψ_N
Baseline	0.66	1.49	0.24	1.00
Spare link A	0.71	1.35	0.28	1.11
Spare link B	0.73	1.30	0.31	1.16
Dual spare links	0.75	1.23	0.34	1.21

DISCUSSION AND CONCLUSION

Multi-state reliability can be represented as finite weighted-language evaluation rather than solely as continuous-time probability flow. The automaton-semiring formulation is especially useful when the analyst needs to reason about admissible sequences of interventions, ambiguity in observed transitions, and competing notions of resilience such as service quality, repair effort, and path diversity. Because the theory is expressed through recurrences, transfer operators, and partial orders, it fits naturally within discrete mathematical structures and supports exact symbolic reasoning in addition to simulation [15–25].

Acknowledgments

The manuscript is a theoretical and simulation-based study.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability

All numerical results are based on synthetic data generated from the reported models and parameter settings.

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