STM JOURNALS

Recent Trends in
Mathematics

Volume 1, Issue 1, 2024
January—April
DOI (Journal): 10.37591/RTM

https://journals.stmjournals.com/rtm

Research RTM

$(\Delta, \text{Ü})$– Convex Structure on Partial *B*-Metric Space Concerning Quasi Contraction and Fixed-Point Results

Umashankar Singh[1,]*Naval Singh[2], Ruchi Singh[3]

Abstract

This work introduces the concept of $(\delta, \text{Ü})$– Convex Partial b-Metric Spaces using convex structure. Motivated by this approach, we demonstrated fixed point results and their uniqueness, as well as quasi contraction, and provided some supporting instances for the established results. Our findings expand prior fixed-point results to a novel concept $(\delta, \text{Ü})$– Convex Partial b-Metric Spaces. To support our theoretical findings, we provide several instances that exemplify the established results, including concrete examples to illustrate the practical applicability and relevance of our work. Our findings significantly extend prior fixed-point results to the novel concept of $(\delta, \text{Ü})$ –Convex Partial b-Metric Spaces. By extending traditional fixed point theory to (δ,Ü)– Convex Partial b-Metric Spaces, this research not only deepens our theoretical understanding but also broadens the scope of applications across diverse fields. In scientific and engineering disciplines, where complex systems and nonlinear dynamics are prevalent, the conceptual framework introduced here offers new tools for analyzing and solving problems. The ability to model and analyze phenomena with convexity considerations opens avenues for more accurate representations of real-world scenarios, potentially leading to more effective solutions and insights This extension not only enhances the understanding of fixed-point theory but also creates new opportunities for its application in more complex and diverse mathematical settings. Consequently, our research advances the field, offering a robust foundation for future studies and potential applications across various scientific and engineering disciplines.

Keywords: $(\delta, \text{Ü})$– Convex Structure, convex partial b-metric space. quasi contraction, fixed point, common fixed point

INTRODUCTION

The Bannach contraction has been generalized in a number of ways and in the literature. Fixed point results about metric space for the mapping satisfying: d(Tx,Ty)≤μ[d(x,Tx)+d(y,Ty)] were established in 1968 by Kannan [1]. for every x,y∈H and μ∈(0,1). In 1970, Takkahashi [2]. presented the idea of convexity and obtained a few fixed-point solutions for non-expansive mapping in metric space. Reich [3]. expanded and generalized the Bannach contraction result in 1971 by proposing a new contraction that, for any x,y∈H and a+b+c≤1, where a, b, and c are nonnegative, satisfied the following formula: d(Tx,Ty)≤ad(x,Tx)+bd(y,Ty)+cd(x,y). Backhtin [4]. first proposed the idea of a b-metric space in 1989, and numerous fixed point tfixed-pointe been proved in b-metric spaces. Czerwik [5]. proved the following outcomes and developed certain fixed-point theorems in b-metric space in 1993.In 1994, Matthews [6]. introduced the notion of partial metric space and proved some fixed-point theorem concerning bannach contraction theorem. Another

***Author for Correspondence**
Umashankar Singh
E-mail: umasirte2021@gmail.com

[1]Associate Professor, Sagar Institute of Research & Technology Excellence, Bhopal, Madhya. Pradesh, India
[2]Professor, Government, SPM Science & Commerce College, Bhopal, Madhya. Pradesh, India
[3]Professor, Department of Mathematics, Pandit S.N. Shukla University, Shahdol, Madhya. Pradesh, India

Received Date: June 25, 2024
Accepted Date: July 04, 2024
Published Date: September 05, 2024

Citation: Umashankar Singh, Naval Singh, Ruchi Singh. (δ,Ü)– Convex Structure on Partial *b*-Metric Space Concerning Quasi Contraction and Fixed-Point Results. Recent Trends in Mathematics. 2024; 1(1): 24–34p.

generalization was convex metric space. Subsequently, Beg [7]. Beg and Abbas [8,9]. Chang, Kim and Jin[10], Ciric[11]. Ding [12]. and many researchers studied fixed point results in convex metric space and convex b-metric spaces. After that fixed point results in partial metric space studied by many researchers. In 2017, Georgescu [13]. studied iterated function systems through partial generalized contractions in the setting of α –complate metric spaces. In 2020, Sawangsup K. Sintunavarat W. and Cho Y.J [14]. established fixed point results on orthogonal complete metric spaces. In 2021, Chen [15]. gave some strong convergence fixed point theorem in the context of convex graphical rectangular b-metric space *(GRbCMS*). In 2021, Karaca et. al [16]. established fixed point theorems for Reich contraction mapping in convex b-metric space satisfying Mann iteration sequence and also weak T-stability. Recently in 2022, D.K Sharma [17]. introduced the notion δ-convex rectangular metric space and proved some fixed point results for rectangular metric space.

Here, in this research paper we study the existence of fixed point by introducing the notion *(δ, Ü)*-convex partial b-metric space and prove fixed point results for different contraction and quote some examples.[18]. These results can further be extended in other metric spaces.

PRELIMINARIES

We Recall Some Basic Definition Which Will Be Utilized in Our Paper:

Definition 2.1[6]: Let H be a nonempty set the function $\Omega_p: H \times H \to [0, \infty]$ is called a partial metric on H if for all $u, v, w \in H$ .then the following conditions are satisfied:

i. $0 \leq \Omega_p(u,u) \leq \Omega_p(u,v)$ $(small\ self\ distance)$
ii. $\Omega_p(u,v) = \Omega_p(v,u)$ $(symmetry)$
iii. $\Omega_p(u,u) = \Omega_p(u,v) = \Omega_c(v,v)\ iff\ u = v$ $(equality)$
iv. $\Omega_p(u,v) \leq \Omega_p(u,w) + \Omega_p(w,v) - \Omega_p(w,w)$ $(triangularity)$

Then the pair (H, Ω_p) is called partial metric spaces on H.

Definition 2.2[6]: Let H be a nonempty set the function $\Omega_{pb}: H \times H \to [0, \infty]$ with $s \geq 1$ is called a partial *b*-metric on H if for all $u, v, w \in H$ .then the following conditions are satisfied:

i. $0 \leq \Omega_{pb}(u,u) \leq \Omega_{pb}(u,v)$ $(small\ self\ distance)$
ii. $\Omega_{pb}(u,v) = \Omega_{pb}(v,u)$ $(symmetry)$
iii. $\Omega_{pb}(u,u) = \Omega_{pb}(u,v) = \Omega_{pb}(v,v)\ iff\ u = v$ $(equality)$
iv. $\Omega_{pb}(u,v) \leq s\left[\Omega_p(u,w) + \Omega_p(w,v) - \Omega_p(w,w)\ \right] (triangularity)$

Then the pair (H, Ω_{pb}) is called partial *b*-metric spaces on H. the number s is called the coefficient of (H, Ω_{pb}).

Definition 2.3[3]: Let a set $C \subseteq R$ and $\delta, \ddot{U} \in [0,1]$ then C is called $(\delta, \ddot{U})$–*convex* if

$\alpha x\delta + (1-\alpha)\ddot{U}y \in C\ for\ all\ x, y \in C\ \ and\ \alpha \in [0,1]$

Definition 2.3[3]: Let a set $C \subseteq R$ and $\delta, \ddot{U} \in [0,1]$ then a mapping $F: C \subset R \to R$ is called $(\delta, \ddot{U})$–*convex* if C be a $(\delta, \ddot{U})$–*convex set and*

$T(\alpha x\delta + (1-\alpha)\ddot{U}y) \leq \alpha\delta T(x) + (1-\alpha)\ddot{U}T(y)\ for\ all\ x, y \in C\ \ and\ \alpha \in [0,1]$

Definition 2.4[3]: Let H be a nonempty set and $I = [0,1]$ .define the function $\Omega_{cb}: H \times H \to [0, \infty]$ and a continuous function $\varpi: H \times H \times J \times I \to H$. then ϖ is called $(\delta, \ddot{U})$–*convex structure on* H *if:*

$$\Gamma\big(t, \varpi(\vartheta, \theta, \varphi; \alpha, \delta, \ddot{U})\big) \leq \alpha\delta\Gamma(t, \vartheta) + (1-\alpha)\ddot{U}\Gamma(t, \theta) \quad for\ all\ t \in H$$

$$and\ \ (\vartheta, \theta, \varphi; \alpha, \delta, \ddot{U}) \in H \times H \times J \times I,\ \ where\ \ J \subseteq I$$

Definition 2.5[3]: Let the function $\varpi: H \times H \times J \times I \rightarrow H$ be a $(\delta,$ Ü $)$*–convex structure on a b-Partial metric space* (H, Ω_{pb}) *and* $I = [0,1]$ *then* (H, Ω_{pb}, ϖ) *is called* $(\delta,$ Ü $)$*–convex b-Partial metric space.*

Definition 2.6[3]: (H, Ω_{pb}, ϖ) be a $(\delta,$ Ü $)$*– Convex b-Partial metric space with a* function $: H \rightarrow H$, for $\vartheta_k \in H$ *and* $\varsigma_k \in [0,1]$ then generalize Mann's iteration sequence (ϑ_k) is defined as :

$$\vartheta_{k+1} = \varpi(\vartheta_k, \Omega\vartheta_k; \varsigma_k, \delta, \text{Ü}), \text{ for all } k \in N$$

Example 2.7: Define $H = R$ *and* $\Omega: H \times H \rightarrow [0, +\infty)$ *be a function defined by* $\Omega(\vartheta, \eta) = |\vartheta - \eta|^{\alpha}$, $\alpha > 1$ and where $\vartheta, \eta \in H$

Let us define the function $\varpi: H \times H \times \frac{1}{2} \times I \rightarrow H$ by $\varpi(\vartheta, \theta, \varphi; \alpha, \delta, \text{Ü}) = \frac{\vartheta + \theta}{2}$

Let $v, \vartheta, \eta \in H$ $\Omega(v, \varpi(\vartheta, \theta, \varphi; \alpha), \delta, \text{Ü}) = \left|\text{v}, \frac{\vartheta + \theta}{2}\right|^{\alpha}$

$$= \left|\text{v} - \frac{\vartheta + \theta}{2}\right|^{\alpha}$$

$$= \left|\frac{2v - \vartheta - \theta}{2}\right|^{\alpha}$$

$$= \left|\frac{(v - \vartheta)}{2} + \frac{(v - \theta)}{2}\right|^{\alpha}$$

$$\leq |2^{-1}(v - \vartheta) + 2^{-1}(v - \theta)|^{\alpha}$$

$$\leq 2^{\alpha-1}[2^{-\alpha}|(v - \vartheta)|^{\alpha} + 2^{-\alpha}|(v - \theta)|^{\alpha}]$$

$$= \alpha\delta\Gamma(v, \vartheta) + (1 - \alpha)\text{Ü}\Gamma(v, \theta)$$

Hence (H, Ω_{pb}, ϖ) be a $(\delta,$ Ü $)$*– Convex b-Partial metric space with* $\alpha = 2^{-1}$ and $\delta, \text{Ü} = 1$. *Now, we take* $\alpha = 2$ then we get

$\Omega(3,5) = |3 - 5|^2 = 4$

$\Omega(3,4) = |3 - 4|^2 = 1$

$\Omega(4,5) = |4 - 5|^2 = 1$

it is does not satisfying the triangle inequality, Indeed

$\Omega(3,5) = 4 > \Omega(3,4) + \Omega(4,5) = 2$

Definition 2.7[6]: Let (H, Ω_{pb}) be a partial *b*-metric space with coefficient s and $\{\vartheta_k\}$ be a sequence in X then

i. *The sequence* $\{\vartheta_k\}$ *is said to be convergent in* (H, Ω_{pb}) *and convergence to* $\vartheta^* \in H$, *if for all* $\epsilon > 0$ *a there exists* $\vartheta_0 \in N$ *such that* $\Omega_{pb}(\vartheta_k, \vartheta^*) < \epsilon$ *for all* $k > k_0$.
ii. *The sequence* $\{\vartheta_k\}$ *is called Cauchy sequence in* (H, Ω_{pb}) *if for all* $\epsilon > 0$ *there exists* $\vartheta_0 \in N$ *such that* $\Omega_{pb}(\vartheta_k, \vartheta_m) < \epsilon$ *for all* $k, m > k_0$.
iii. (H, Ω_{pb}) *is said to be a complete partial b-metric space if every Cauchy sequence is Convergent in H.*

Now the following theorems prove by using (δ, Ü)-convex Partial b-metric space

Our Main Results

Theorem 3.1: Let (H_C, Γ, ϖ) be a complete $(\delta, Ü)$-*convex Partial b-metric space* with constants $and\ \ \Omega: H_C \to H_C$ be a self mapping defined as:

$$\Gamma(\Omega\vartheta, \Omega\eta) \leq \Lambda\ Max\{\Gamma(\vartheta,\eta), \Gamma(\vartheta,\Omega\vartheta), \Gamma(\eta,\Omega\eta\ )\}\ \ , for\ \vartheta,\eta \in H_C$$

and for some $\Lambda \in \left[0, \frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Gamma(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty\ \ and$

$define\ \vartheta_k = \varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, Ü)$ for $k \in$ N and $0 < \varsigma_{k-1} < \frac{1}{s^2}$, $\delta, Ü \leq [0,1)$.

Then $\Omega\ has\ a\ unique\ fixed\ point\ of\ H_C$.

Proof: for $k \in$ N , we have

$$\Gamma(\vartheta_k, \vartheta_{k+1}) = \Gamma\big(\vartheta_k, \varpi(\vartheta_k, \Omega\vartheta_k; \varsigma_k, \delta, Ü)\big) = \varsigma_k\ \delta\Gamma(\vartheta_k, \vartheta_k) + (1 - \varsigma_k\ )Ü\Gamma(\vartheta_k, \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \vartheta_{k+1}) \leq (1 - \varsigma_k\ )Ü\Gamma(\vartheta_k, \Omega\vartheta_k).$$

Now,

$$\Gamma(\vartheta_k, \Omega\vartheta_k) = \Gamma(\varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, Ü), \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \varsigma_{k-1}\ \delta\Gamma(\vartheta_{k-1}, \Omega\vartheta_k) + (1 - \varsigma_{k-1})Ü\Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \varsigma_{k-1}\ s\delta[\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k) - \Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_{k-1})] + (1 - \varsigma_k\ )Ü\Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$

$$\leq s\delta\varsigma_{k-1}\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1 - \varsigma_{k-1})Ü\}\Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$

$$\leq s\delta\varsigma_{k-1}\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

$$+\{s\delta\varsigma_{k-1} + (1 - \varsigma_{k-1})Ü\}\Lambda\ Max\{\Gamma(\vartheta_{k-1}, \vartheta_k), \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Gamma(\vartheta_k, \Omega\vartheta_k)\}$$

$$\leq s\delta\varsigma_{k-1}\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

$$+\{s\delta\varsigma_{k-1} + (1 - \varsigma_{k-1})Ü\}\Lambda\ Max\begin{Bmatrix}(1 - \varsigma_{k-1})Ü\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Gamma(\vartheta_k, \Omega\vartheta_k)\end{Bmatrix}$$

Where $\beta = (1 - \varsigma_{k-1})Ü \leq 1$

$$\leq s\delta\varsigma_{k-1}\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

$$+\{s\delta\varsigma_{k-1} + \beta\}\Lambda\ Max\begin{Bmatrix}\beta\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Gamma(\vartheta_k, \Omega\vartheta_k)\end{Bmatrix}$$

There are two possibilities:

Case (i) if $Max\begin{Bmatrix}\beta\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Gamma(\vartheta_k, \Omega\vartheta_k)\end{Bmatrix} = \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Then

$$\leq s\delta\varsigma_{k-1}\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

$$\leq \{s\delta\varsigma_{k-1} + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\}\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

Put $K_1 = \{s\delta\varsigma_{k-1} + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\} \leq 1$

$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq K_1\, \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Since $0 < \varsigma_{k-1} < \frac{1}{2}, \quad \delta, Ü \leq 1, \Lambda \in \left[0, \frac{1}{2}\right)$ $then\ K_1 \leq 1$.

Case (ii) if $Max\left\{\begin{matrix} \beta\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Gamma(\vartheta_k, \Omega\vartheta_k) \end{matrix}\right\} = \Gamma(\vartheta_k, \Omega\vartheta_k)$

Then

$\leq s\delta\varsigma_{k-1}\, \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\, \Gamma(\vartheta_k, \Omega\vartheta_k)$

$[1 - \{s\delta\varsigma_{k-1} + \beta\}\Lambda]\Gamma(\vartheta_k, \Omega\vartheta_k) \leq s\delta\varsigma_{k-1}\, \Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \dfrac{s\delta\varsigma_{k-1}}{[1 - \{s\delta\varsigma_{k-1} + \beta\}\Lambda]}\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Put $K_2 = \frac{s\delta\varsigma_{k-1}}{[1 - \{s\delta\varsigma_{k-1} + \beta\}\Lambda]} \leq 1$

$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq K_2\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Since $0 < \varsigma_{k-1} < \frac{1}{2}, \quad \delta, Ü \leq 1, \Lambda \in \left[0, \frac{1}{2}\right)$ $then\ K_1 \leq 1$.

Let $K = Max\{K_1, K_2\}$ then
$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq K\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Since $\delta, Ü, \varsigma_{k-1} \in \left[0, \frac{1}{2}\right)$ $and\ K \leq 1$, $\Gamma(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ then applying $k \to \infty$ then we get $\lim_{k\to\infty} \Gamma(\vartheta_k, \Omega\vartheta_k) = 0$ and since, $\Gamma(\vartheta_k, \vartheta_{k+1}) \leq (1 - \varsigma_k)Ü\Gamma(\vartheta_k, \Omega\vartheta_k) < \Gamma(\vartheta_k, \Omega\vartheta_k)$

Thus, $\lim_{k\to\infty} \Gamma(\vartheta_k, \vartheta_{k+1}) = 0$

Now we show that $\{\vartheta_k\}$ is a Cauchy sequence. Suppose $\{\vartheta_k\}$ is not a Cauchy sequence and let two subsequences $\{\vartheta_{p_t}\}$ and $\{\vartheta_{q_t}\}$ of $\{\vartheta_k\}$ such that $q_t > p_t > t$ satisfying $\Gamma\left(\vartheta_{p_t}, \vartheta_{q_t}\right) \geq \epsilon\ and\ \Gamma\left(\vartheta_{p_t-1}, \vartheta_{q_t}\right) < \epsilon$ then by $triangularity$ property

$$\epsilon \leq \Gamma\left(\vartheta_{p_t}, \vartheta_{q_t}\right) \leq s\left[\Gamma\left(\vartheta_{p_t}, \vartheta_{q_t+1}\right) + \Gamma\left(\vartheta_{q_t+1}, \vartheta_{q_t}\right) - \Gamma\left(\vartheta_{q_t+1}, \vartheta_{q_{t+1}}\right)\right]$$

Taking $t \to \infty$ and applying above values then we get

$$\limsup_{t\to\infty} \Gamma\left(\vartheta_{p_t}, \vartheta_{q_t}\right) \leq \epsilon$$

Now, $\Gamma\left(\vartheta_{p_t}, \vartheta_{q_t+1}\right) = \Gamma\left(\left(\varpi\left(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1}; \varsigma_{p_t-1}, \delta, Ü\right), \vartheta_{p_t+1}\right)\right)$

$\leq \delta\varsigma_{p_t-1}\, \Gamma\left(\vartheta_{p_t-1}, \vartheta_{p_t+1}\right) + \left(1 - \varsigma_{p_t-1}\right)Ü\Gamma\left(\Omega\vartheta_{p_t-1}, \vartheta_{p_t+1}\right)$

$\leq \delta\varsigma_{p_t-1}\, \Gamma\left(\vartheta_{p_t-1}, \vartheta_{p_t+1}\right)$

$+(1 - \varsigma_{p_t-1})Üs\left[\Gamma\left(\Omega\vartheta_{p_t-1}, \vartheta_{p_t}\right) + \Gamma(\vartheta_{p_t}, \vartheta_{p_t+1})\right]$

$\left\{1 - \left(1 - \varsigma_{p_t-1}\right)Üs\right\}\Gamma\left(\vartheta_{p_t}, \vartheta_{p_t+1}\right) \leq \delta\varsigma_{p_t-1}\, \Gamma\left(\vartheta_{p_t-1}, \vartheta_{p_t+1}\right)$

$+(1$
$-\varsigma_{p_{t-1}})$Üs $\Gamma\left(\Omega\vartheta_{p_{t-1}},\vartheta_{p_t}\right)$

Therefore letting $t\to\infty$ then we obtain $\lim_{t\to\infty} Sup\,\Gamma\left(\vartheta_{p_t},\vartheta_{p_{t+1}}\right)\le\epsilon$, which is contradiction. Thus $\{\vartheta_k\}$ is a Cauchy sequenceH_C. By completeness of H_C there exists $\vartheta^*\in H_C$ such that $\lim_{t\to\infty}\Gamma\left(\vartheta_p,\vartheta^*\right)=0$.

Now we claim ϑ^* is a fixed point of Ω.

$\Gamma(\vartheta^*,\Omega\vartheta^*)\le s\left[\Gamma\left(\vartheta^*,\vartheta_\mathrm{p}\right)+\Gamma\left(\vartheta_\mathrm{p},\Omega\vartheta^*\right)-\Gamma(\Omega\vartheta^*,\Omega\vartheta^*)\right]$

$\Gamma(\vartheta^*,\Omega\vartheta^*)\le s^2\left[\Gamma\left(\vartheta^*,\vartheta_\mathrm{p}\right)+\Gamma\left(\vartheta_\mathrm{p},\Omega\vartheta_p\right)\right]+s\left[\Gamma\left(\Omega\vartheta_p,\Omega\vartheta^*\right)\right]$

$\Gamma(\vartheta^*,\Omega\vartheta^*)\le s^2\left[\Gamma\left(\vartheta^*,\vartheta_\mathrm{p}\right)+\Gamma\left(\vartheta_\mathrm{p},\Omega\vartheta_p\right)\right]+\Lambda sMax\left\{\Gamma\left(\vartheta_p,\vartheta^*\right),\Gamma\left(\vartheta_p,\Omega\vartheta_p\right),\Gamma(\vartheta^*,\Omega\vartheta^*)\right\}$

$\Gamma(\vartheta^*,\Omega\vartheta^*)\le s^2\left[\Gamma\left(\vartheta^*,\vartheta_\mathrm{p}\right)+\Gamma\left(\vartheta_\mathrm{p},\Omega\vartheta_p\right)\right]+\Lambda sMax\left\{\Gamma\left(\vartheta_p,\vartheta^*\right),\Gamma\left(\vartheta_p,\Omega\vartheta_p\right),\Gamma(\vartheta^*,\Omega\vartheta^*)\right\}$

Letting $t\to\infty$ *then we get*

$\Gamma(\vartheta^*,\Omega\vartheta^*)\le\Lambda sMax\{\Gamma(\vartheta^*,\Omega\vartheta^*),s\Gamma(\vartheta^*,\Omega\vartheta^*)\}$

$\Gamma(\vartheta^*,\Omega\vartheta^*)\le\Lambda s^2\Gamma(\vartheta^*,\Omega\vartheta^*)$

So $\Gamma(\vartheta^*,\Omega\vartheta^*)=0\Rightarrow\vartheta^*=\Omega\vartheta^*$

Hence ϑ^* is a fixed point of of Ω.

Uniqueness: Suppose Θ is another fixed point of Ω then we have $\Omega\vartheta^*=\vartheta^*$ *and* $\Omega\Theta=\Theta$.

Now $\Gamma(\vartheta^*,\Theta)=\Gamma(\Omega\vartheta^*,\Omega\Theta)$

$\le\Lambda Max\{\Gamma(\vartheta^*,\Theta),\Gamma(\vartheta^*,\Omega\vartheta^*),\Gamma(\Theta,\Omega\Theta)\}\le\Lambda Max\{\Gamma(\vartheta^*,\Theta),\Gamma(\vartheta^*,\vartheta^*),\Gamma(\Theta,\Theta)\}$

$\Gamma(\vartheta^*,\Theta)\le\Lambda\Gamma(\vartheta^*,\Theta)$

$(1-\Lambda)\,\Gamma(\vartheta^*,\Theta)\le 0$

$(1-\Lambda)\ne 0,\ \Gamma(\vartheta^*,\Theta)=0\Rightarrow\vartheta^*=\Theta$

Hence proof.

Theorem 3.2: Let (H_C,Γ,ϖ) be a complete $(\delta,\text{Ü})$-*convex Partial b-metric space* with constants $s\ge 1$ *and* $\Omega: H_C\to H_C$ be a quasi contraction mapping defined as: [19].

$\Gamma(\Omega\vartheta,\Omega\eta)\le\Lambda\,Max\{\Gamma(\vartheta,\eta),\Gamma(\vartheta,\Omega\vartheta),\Gamma(\eta,\Omega\eta),\Gamma(\vartheta,\Omega\eta),\Gamma(\eta,\Omega\vartheta)\}$, $for\ \vartheta,\eta\in H_C$

and for some $\Lambda\in\left[0,\frac{1}{2}\right)$. Take $\vartheta_0\in H_C$ such that $\Gamma(\vartheta_0,\Omega\vartheta_0)=\mathfrak{M}<\infty$ *and*

define $\vartheta_k=\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\text{Ü})$ for $k\in$ N and $0<\varsigma_{k-1}<\frac{1}{s^2},\delta,\text{Ü}\le[0,1)$.

Then Ω *has a unique fixed point of* H_C.

Proof: for $k\in$ N , we have

$\Gamma(\vartheta_k,\vartheta_{k+1})=\Gamma\left(\vartheta_k,\varpi(\vartheta_k,\Omega\vartheta_k;\varsigma_k,\delta,\text{Ü})\right)=\varsigma_k\,\delta\Gamma(\vartheta_k,\vartheta_k)+(1-\varsigma_k)\text{Ü}\Gamma(\vartheta_k,\Omega\vartheta_k)$

$\Gamma(\vartheta_k,\vartheta_{k+1})\le(1-\varsigma_k)\text{Ü}\Gamma(\vartheta_k,\Omega\vartheta_k)$.

Now,

$\Gamma(\vartheta_k,\Omega\vartheta_k) = \Gamma(\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U}),\Omega\vartheta_k)$

$\Gamma(\vartheta_k,\Omega\vartheta_k) \le \varsigma_{k-1}\ \delta\Gamma(\vartheta_{k-1},\Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Gamma(\Omega\vartheta_{k-1},\Omega\vartheta_k)$

$$\Gamma(\vartheta_k,\Omega\vartheta_k) \le \varsigma_{k-1}\ s\delta[\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Gamma(\Omega\vartheta_{k-1},\Omega\vartheta_k) - \Gamma(\Omega\vartheta_{k-1},\Omega\vartheta_{k-1})] + (1-\varsigma_k)\ddot{U}\Gamma(\Omega\vartheta_{k-1},\Omega\vartheta_k)$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Gamma(\Omega\vartheta_{k-1},\Omega\vartheta_k)$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\begin{Bmatrix}\Gamma(\vartheta_{k-1},\vartheta_k),\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Gamma(\vartheta_k,\Omega\vartheta_k),\Gamma(\vartheta_{k-1},\Omega\vartheta_k),\Gamma(\vartheta_k,\Omega\vartheta_{k-1})\end{Bmatrix}$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\begin{Bmatrix}(1-\varsigma_{k-1})\ddot{U}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Gamma(\vartheta_k,\Omega\vartheta_k),\Gamma(\vartheta_{k-1},\Omega\vartheta_k),\Gamma(\vartheta_k,\Omega\vartheta_{k-1})\end{Bmatrix}$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\begin{Bmatrix}(1-\varsigma_{k-1})\ddot{U}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Gamma(\vartheta_k,\Omega\vartheta_k),\{\Gamma(\vartheta_{k-1},\vartheta_k) + \Gamma(\vartheta_k,\Omega\vartheta_k)\},\\ \Gamma(\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U}),\Omega\vartheta_{k-1})\end{Bmatrix}$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\begin{Bmatrix}(1-\varsigma_{k-1})\ddot{U}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Gamma(\vartheta_k,\Omega\vartheta_k),\{(1-\varsigma_{k-1})\ddot{U}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Gamma(\vartheta_k,\Omega\vartheta_k)\},\\ \Gamma(\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U}),\Omega\vartheta_{k-1})\end{Bmatrix}$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\begin{Bmatrix}(1-\varsigma_{k-1})\ddot{U}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Gamma(\vartheta_k,\Omega\vartheta_k),\{(1-\varsigma_{k-1})\ddot{U}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Gamma(\vartheta_k,\Omega\vartheta_k)\},\\ \varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + (1-\varsigma_{k-1})\Gamma(\Omega\vartheta_{k-1},\Omega\vartheta_{k-1})\end{Bmatrix}$$

$$\le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \xi\}\Lambda\, Max\begin{Bmatrix}\xi\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Gamma(\vartheta_k,\Omega\vartheta_k),\\ \{\xi\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Gamma(\vartheta_k,\Omega\vartheta_k)\},\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1})\end{Bmatrix}$$

Where $\xi = (1-\varsigma_{k-1})\ddot{U} \le 1$

$$\Gamma(\vartheta_k,\Omega\vartheta_k) \le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \xi\}\Lambda\, Max\begin{Bmatrix}\xi\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \{\xi\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Gamma(\vartheta_k,\Omega\vartheta_k)\}\end{Bmatrix}$$

$$\Gamma(\vartheta_k,\Omega\vartheta_k) \le s\delta\varsigma_{k-1}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \xi\}\Lambda\,\xi\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Gamma(\vartheta_k,\Omega\vartheta_k)$$

$$[1-\{s\delta\varsigma_{k-1} + \xi\}\Lambda]\Gamma(\vartheta_k,\Omega\vartheta_k) \le [s\delta\varsigma_{k-1} + \{s\delta\varsigma_{k-1} + \xi\}\Lambda\xi]\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

$$\Gamma(\vartheta_k,\Omega\vartheta_k) \le \frac{[s\delta\varsigma_{k-1} + \{s\delta\varsigma_{k-1} + \xi\}\Lambda\xi]}{[1-\{s\delta\varsigma_{k-1} + \xi\}\Lambda]}\Gamma(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

Since $\delta,\ddot{U} \in [0,1), \Lambda \in \left[0,\frac{1}{2}\right)$, $0 < \varsigma_{k-1} < \frac{1}{s^2}, and\ \ \xi \le 1, \Gamma(\vartheta_0,\Omega\vartheta_0) = \mathfrak{M} < \infty$. then applying $k \to \infty$ then we get $\lim_{k\to\infty}\Gamma(\vartheta_k,\Omega\vartheta_k) = 0$ and

since, $\Gamma(\vartheta_k,\vartheta_{k+1}) \le (1-\varsigma_k)\ddot{U}\Gamma(\vartheta_k,\Omega\vartheta_k) < \Gamma(\vartheta_k,\Omega\vartheta_k)$
Thus, $\lim_{k\to\infty}\Gamma(\vartheta_k,\vartheta_{k+1}) = 0$

Now we show that $\{\vartheta_k\}$ is a Cauchy sequence. Suppose $\{\vartheta_k\}$ is not a Cauchy sequence and let two subsequences $\{\vartheta_{p_t}\}$ and $\{\vartheta_{q_t}\}$ of $\{\vartheta_k\}$ such that $q_t > p_t > t$ satisfying $\Gamma(\vartheta_{p_t}, \vartheta_{q_t}) \geq \epsilon$ and $\Gamma(\vartheta_{p_t-1}, \vartheta_{q_t}) < \epsilon$ then by $triangularity$ property

$$\epsilon \leq \Gamma(\vartheta_{p_t}, \vartheta_{q_t}) \leq s\left[\Gamma(\vartheta_{p_t}, \vartheta_{q_t+1}) + \Gamma(\vartheta_{q_t+1}, \vartheta_{q_t}) - \Gamma(\vartheta_{q_t+1}, \vartheta_{q_{t+1}})\right]$$

Taking $t \to \infty$ and applying above values then we get

$$\limsup_{t\to\infty} \Gamma(\vartheta_{p_t}, \vartheta_{q_t}) \leq \epsilon$$

Now, $\Gamma(\vartheta_{p_t}, \vartheta_{q_t+1}) = \Gamma\left((\varpi(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1}; \varsigma_{p_t-1}, \delta, \ddot{U}), \vartheta_{p_t+1})\right)$

$$\leq \delta\varsigma_{p_t-1}\,\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t+1}) + (1-\varsigma_{p_t-1})\ddot{U}\Gamma(\Omega\vartheta_{p_t-1}, \vartheta_{p_t+1})$$

$$\leq \delta\varsigma_{p_t-1}\,\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t+1}) + (1-\varsigma_{p_t-1})\ddot{U}s\left[\Gamma(\Omega\vartheta_{p_t-1}, \Omega\vartheta_{p_t+1}) + \Gamma(\Omega\vartheta_{p_t+1}, \vartheta_{p_t+1})\right]$$

$$\leq \delta\varsigma_{p_t-1}\,\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t+1}) + (1-\varsigma_{p_t-1})\ddot{U}s\Gamma(\Omega\vartheta_{p_t+1}, \vartheta_{p_t+1}) + (1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\,Max\left\{\begin{matrix}\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t+1}), \Gamma(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1}),\\ \Gamma(\vartheta_{p_t+1}, \Omega\vartheta_{p_t+1}), \Gamma(\vartheta_{p_t-1}, \Omega\vartheta_{p_t+1}),\\ \Gamma(\vartheta_{p_t+1}, \Omega\vartheta_{p_t-1})\end{matrix}\right\}$$

$$\leq \delta\varsigma_{p_t-1}\,s\left[\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t}) + \Gamma(\vartheta_{p_t}, \vartheta_{p_t+1})\right] + (1-\varsigma_{p_t-1})\ddot{U}s\Gamma(\Omega\vartheta_{p_t+1}, \vartheta_{p_t+1}) + (1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\,Max\left\{\begin{matrix}s\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t}) + s\Gamma(\vartheta_{p_t}, \vartheta_{p_t+1}),\\ \Gamma(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1}), \Gamma(\vartheta_{p_t+1}, \Omega\vartheta_{p_t+1}),\\ s\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t+1}) + s\Gamma(\vartheta_{p_t+1}, \Omega\vartheta_{p_t+1}),\\ s\Gamma(\vartheta_{p_t+1}, \vartheta_{p_t-1}) + s\Gamma(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1})\end{matrix}\right\}$$

$$\leq \delta\varsigma_{p_t-1}\,s\left[\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t}) + \Gamma(\vartheta_{p_t}, \vartheta_{p_t+1})\right] + (1-\varsigma_{p_t-1})\ddot{U}s\Gamma(\Omega\vartheta_{p_t+1}, \vartheta_{p_t+1}) + (1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\,Max\left\{\begin{matrix}s\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t}) + s\Gamma(\vartheta_{p_t}, \vartheta_{p_t+1}),\\ \Gamma(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1}), \Gamma(\vartheta_{p_t+1}, \Omega\vartheta_{p_t+1}),\\ s\left\{s\left[\Gamma(\vartheta_{p_t-1}, \vartheta_{p_t}) + \Gamma(\vartheta_{p_t}, \vartheta_{p_t+1})\right]\right\} + s\Gamma(\vartheta_{p_t+1}, \Omega\vartheta_{p_t+1}),\\ s\left\{s\left[\Gamma(\vartheta_{p_t+1}, \vartheta_{p_t}) + \Gamma(\vartheta_{p_t}, \vartheta_{p_t-1})\right]\right\} + s\Gamma(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1})\end{matrix}\right\}$$

$$\leq \delta\varsigma_{p_t-1}\,s\epsilon + (1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\,Max\{s\epsilon, s^2\epsilon\}$$

$$\leq \delta\varsigma_{p_t-1}\,s\epsilon + (1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda s^2\epsilon$$

Letting $t \to \infty$ then we get $\lim_{t\to\infty} sup\, \Gamma(\vartheta_{p_t}, \vartheta_{q_t+1}) < \epsilon$. This is contradiction. Thus $\{\vartheta_{p_t}\}$ being Cauchy sequence in H. By completeness of H , there exist $\vartheta^* \in H$ such that $\lim_{t\to\infty} \Gamma(\vartheta_p, \vartheta^*) = 0$.

Now we shall prove ϑ^* is a fixed point of of Ω for this

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \leq s\left[\Gamma(\vartheta^*, \vartheta_p) + \Gamma(\vartheta_p, \Omega\vartheta^*) - \Gamma(\Omega\vartheta^*, \Omega\vartheta^*)\right]$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \leq s\left[\Gamma(\vartheta^*, \vartheta_p) + \Gamma(\vartheta_p, \Omega\vartheta_p)\right] + s^2\left[\Gamma(\Omega\vartheta_p, \Omega\vartheta^*)\right]$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \leq s\left[\Gamma(\vartheta^*, \vartheta_p) + \Gamma(\vartheta_p, \Omega\vartheta_p)\right] + \Lambda s^2 Max\{\Gamma(\vartheta_p, \vartheta^*), \Gamma(\vartheta_p, \Omega\vartheta_p), \Gamma(\vartheta^*, \Omega\vartheta^*), \Gamma(\vartheta_p, \Omega\vartheta^*), \Gamma(\vartheta^*, \Omega\vartheta_p)\}$$

$\Gamma(\vartheta^*, \Omega\vartheta^*) \leq$

$$s\left[\Gamma(\vartheta^*, \vartheta_p) + \Gamma(\vartheta_p, \Omega\vartheta_p)\right] + \Lambda s^2 Max\left\{\begin{matrix}\Gamma(\vartheta_p, \vartheta^*), \Gamma(\vartheta_p, \Omega\vartheta_p), \Gamma(\vartheta^*, \Omega\vartheta^*), \\ s\Gamma(\vartheta_p, \vartheta^*) + s\Gamma(\vartheta^*, \Omega\vartheta^*), s\Gamma(\vartheta^*, \vartheta_p) + s\Gamma(\vartheta_p, \Omega\vartheta_p)\end{matrix}\right\}$$

Letting $t \to \infty$ $then\ we\ get$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \leq \Lambda s^2 Max\{\Gamma(\vartheta^*, \Omega\vartheta^*), s\Gamma(\vartheta^*, \Omega\vartheta^*)\}$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \leq \Lambda s^2 \Gamma(\vartheta^*, \Omega\vartheta^*)$$

So $\Gamma(\vartheta^*, \Omega\vartheta^*) = 0 \Rightarrow \vartheta^* = \Omega\vartheta^*$

Hence ϑ^* is a fixed point of of Ω.

Uniqueness: Suppose Θ is another fixed point of Ω then we have $\Omega\vartheta^* = \vartheta^*$ and $\Omega\Theta = \Theta$.
Now $\Gamma(\vartheta^*, \Theta) = \Gamma(\Omega\vartheta^*, \Omega\Theta)$

$$\leq \Lambda Max\{\Gamma(\vartheta^*, \Theta), \Gamma(\vartheta^*, \Omega\vartheta^*), \Gamma(\Theta, \Omega\Theta), \Gamma(\vartheta^*, \Omega\Theta), \Gamma(\Theta, \Omega\vartheta^*)\}$$

$$\leq \Lambda Max\{\Gamma(\vartheta^*, \Theta), \Gamma(\vartheta^*, \vartheta^*), \Gamma(\Theta, \Theta), \Gamma(\vartheta^*, \Theta), \Gamma(\Theta, \vartheta^*)\}$$

$$\Gamma(\vartheta^*, \Theta) \leq \Lambda \Gamma(\vartheta^*, \Theta)$$

$$(1-\Lambda)\,\Gamma(\vartheta^*, \Theta) \leq 0$$

$(1-\Lambda) \neq 0,\ \Gamma(\vartheta^*, \Theta) = 0 \Rightarrow \vartheta^* = \Theta$

Hence proof.

Theorem 3.3: Let (H_C, Γ, ϖ) be a complete $(\delta, \ddot{U})$-*convex Partial b-metric space* with constants $s \geq 1$ and $\Omega: H_C \to H_C$ be a kannan type contraction mapping defined as:[20]

$$\Gamma(\Omega\vartheta, \Omega\eta) \leq \Lambda\{\Gamma(\vartheta, \Omega\vartheta) + \Gamma(\eta, \Omega\eta)\}\ ,\ for\ \vartheta, \eta \in H_C$$

and for some $\Lambda \in \left[0, \frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Gamma(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ and $define\ \vartheta_k = \varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U})$ for $k \in$ N and $0 < \varsigma_{k-1} < \frac{1}{s}$, and $\delta, \ddot{U} \leq [0,1)$. Then $\Omega\ has\ a\ unique\ fixed\ point\ of\ H_C$.

Proof: for $k \in$ N , we have

$$\Gamma(\vartheta_k, \vartheta_{k+1}) = \Gamma\left(\vartheta_k, \varpi(\vartheta_k, \Omega\vartheta_k; \varsigma_k, \delta, \ddot{U})\right) = \varsigma_k\ \delta\Gamma(\vartheta_k, \vartheta_k) + (1-\varsigma_k)\ddot{U}\Gamma(\vartheta_k, \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \vartheta_{k+1}) \leq (1-\varsigma_k)\ddot{U}\Gamma(\vartheta_k, \Omega\vartheta_k).$$

Now,

$$\Gamma(\vartheta_k, \Omega\vartheta_k) = \Gamma(\varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U}), \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \varsigma_{k-1}\ \delta\Gamma(\vartheta_{k-1}, \Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \varsigma_{k-1}\ s\delta[\Gamma(\vartheta_{k-1}, \vartheta_k) + \Gamma(\vartheta_k, \Omega\vartheta_k) - \Gamma(\Omega\vartheta_k, \Omega\vartheta_k)] + (1-\varsigma_{k-1})\ddot{U}\Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \varsigma_{k-1}\ s\delta[\Gamma(\vartheta_{k-1}, \vartheta_k) + \Gamma(\vartheta_k, \Omega\vartheta_k)] + (1-\varsigma_{k-1})\ddot{U}\Gamma(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$

$$\Gamma(\vartheta_k, \Omega\vartheta_k) \leq \varsigma_{k-1}\ s\delta(1-\varsigma_{k-1})\Gamma(\vartheta_{k-1}, \Omega\,\vartheta_{k-1}) + \varsigma_{k-1}\ s\delta\Gamma(\vartheta_k, \Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Lambda\{\Gamma(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \Gamma(\vartheta_k, \Omega\vartheta_k)\}$$

$$[1-\varsigma_{k-1}\ s\delta - (1-\varsigma_k)\ddot{U}\Lambda]\Gamma(\vartheta_k, \Omega\vartheta_k) \leq (1-\varsigma_{k-1})[\varsigma_{k-1}\ s\delta + \ddot{U}\Lambda]\Gamma(\vartheta_{k-1}, \Omega\,\vartheta_{k-1})$$

$\Gamma(\vartheta_k, \Omega\vartheta_k) \le \frac{(1-\varsigma_{k-1})[\varsigma_{k-1} s\delta+\ddot{U}\Lambda]}{[1-\varsigma_{k-1} s\delta-(1-\varsigma_k)\ddot{U}\Lambda]} \Gamma(\vartheta_{k-1}, \Omega\, \vartheta_{k-1})$

Put $j = \frac{(1-\varsigma_{k-1})[\varsigma_{k-1} s\delta+\ddot{U}\Lambda]}{[1-\varsigma_{k-1} s\delta-(1-\varsigma_k)\ddot{U}\Lambda]} \le 1.$

$\Gamma(\vartheta_k, \Omega\vartheta_k) \le j\Gamma(\vartheta_{k-1}, \Omega\, \vartheta_{k-1})$

Since $\delta, \ddot{U} \in [0,1)$ *and* $j \le 1$, $\Lambda \in \left[0, \frac{1}{2}\right)$, $0 < \varsigma_{k-1} < \frac{1}{s}$, *and* $\Gamma(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ then gives $\Gamma(\vartheta_k, \Omega\vartheta_k)$ is non increasing sequence of positive reals. By Theorem (3.1) we follows $\{\vartheta_k\}$ is a Cauchy sequence. Now we shall prove ϑ^* is a fixed point of of Ω for this

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \le s\left[\Gamma(\vartheta^*, \vartheta_p) + \Gamma(\vartheta_p, \Omega\vartheta^*) - \Gamma(\Omega\vartheta^*, \Omega\vartheta^*)\right]$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \le s\left[\Gamma(\vartheta^*, \vartheta_p) + \Gamma(\vartheta_p, \Omega\vartheta^*)\right]$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \le s\Gamma(\vartheta^*, \vartheta_p) + s^2\left[\Gamma(\vartheta_p, \Omega\vartheta_p) + \Gamma(\Omega\vartheta_p, \Omega\vartheta^*)\right]$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \le s\Gamma(\vartheta^*, \vartheta_p) + s^2\Gamma(\vartheta_p, \Omega\vartheta_p) + s^2\Lambda\left[\Gamma(\vartheta_p, \Omega\vartheta_p) + \Gamma(\vartheta^*, \Omega\vartheta^*)\right]$$

$$[1 - s^2\Lambda]\Gamma(\vartheta^*, \Omega\vartheta^*) \le s\Gamma(\vartheta^*, \vartheta_p) + (1 + s^2)\Gamma(\vartheta_p, \Omega\vartheta_p)$$

$$\Gamma(\vartheta^*, \Omega\vartheta^*) \le \frac{s}{[1 - s^2\Lambda]}\Gamma(\vartheta^*, \vartheta_p) + \frac{(1 + s^2)}{[1 - s^2\Lambda]}\Gamma(\vartheta_p, \Omega\vartheta_p)$$

As $p \to \infty$, *we get* $\lim_{p\to\infty} \Gamma(\vartheta^*, \Omega\vartheta^*) = 0$

So $\Gamma(\vartheta^*, \Omega\vartheta^*) = 0 \Rightarrow \vartheta^* = \Omega\vartheta^*$

Hence ϑ^* is a fixed point of of Ω.

Uniqueness: Suppose Θ is another fixed point of Ω then we have $\Omega\vartheta^* = \vartheta^*$ *and* $\Omega\Theta = \Theta$.
Now $\Gamma(\vartheta^*, \Theta) = \Gamma(\Omega\vartheta^*, \Omega\Theta)$

$\le \Lambda\{\Gamma(\vartheta^*, \Omega\vartheta^*) + \Gamma(\Theta, \Omega\Theta)\} \le \Lambda\{\Gamma(\vartheta^*, \vartheta^*) + \Gamma(\Theta, \Theta)\}$

$\Gamma(\vartheta^*, \Theta) = 0 \Rightarrow \vartheta^* = \Theta$

Hence proof.

Theorem 3.4: Let (H_C, Γ, ϖ) be a complete $(\delta, \ddot{U})$-*convex Partial b-metric space* with constants $s \ge 1$ *and* $\Omega: H_C \to H_C$ be a Reich type contraction mapping defined as:

$\Gamma(\Omega\vartheta, \Omega\eta) \le \theta\Gamma(\vartheta, \eta) + \Lambda\{\Gamma(\vartheta, \Omega\vartheta) + \Gamma(\eta, \Omega\eta)\}$, *for* $\vartheta, \eta \in H_C$

and for some $\Lambda \in \left[0, \frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Gamma(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ *and*

define $\vartheta_k = \varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U})$ for $k \in$ N and $0 < \varsigma_{k-1} < \frac{1}{s}$, *and* $\delta, \ddot{U} \le [0,1)$. Then Ω *has a unique fixed point of* H_C.

Proof: similar prove as Theorem 3.3.

CONCLUSION

In this research paper we studied study the existence of fixed point by introducing the notion *(δ,Ü)*-convex structure and prove fixed point results for different types of contraction in concerning of partial b-metric space and quote some examples . These results can further be extended in other metric spaces.

REFERENCES

1. Kannan R. Some results on fixed points. Bull. Cal. Math. Soc. 1968; 60:71-6.
2. Takahashi W. A convexity in metric space and nonexpansive mappings, I. InKodai mathematical seminar reports 1970 (Vol. 22, No. 2, pp. 142-149). Department of Mathematics, Tokyo Institute of Technology.
3. Reich S. Some remarks concerning contraction mappings. Canadian Mathematical Bulletin. 1971 Jan;14(1):121-4.
4. Bakhtin I. The contraction mapping principle in quasimetric spaces. Functional analysis. 1989; 30:26-37.
5. Czerwik S. Contraction mappings in $ b $-metric spaces. Acta mathematica et informatica universitatis ostraviensis. 1993;1(1):5-11.
6. Matthews SG. Partial metric topology. Annals of the New York Academy of Sciences. 1994 Nov;728(1):183-97.
7. Beg I, Abbas MU. Common fixed points and best approximation in convex metric spaces. Soochow Journal of Mathematics. 2007 Oct;33(4):729.
8. Beg I, Abbas M. Fixed point theorems for weakly inward multivalued maps on a convex metric space. Demonstratio Mathematica. 2006 Jan 1;39(1):149-60.
9. Beg I. An iteration scheme for asymptotically nonexpansive mappings on uniformly convex metric spaces. InNonlinear Anal. Forum 2001 (Vol. 6, No. 1, pp. 27-34).
10. Kim JK, Kim KH, Kim KS. Three-Step Iterative Sequences with Errors for Asymptotically Quasi-Nonexpansive Mappings in Convex Metric Spaces (Nonlinear Analysis and Convex Analysis) 2004; 1365:156-65.
11. Chen L, Li C, Kaczmarek R, Zhao Y. Several fixed-point theorems in convex b-metric spaces and applications. Mathematics. 2020 Feb 14;8(2):242.
12. Ćirić L. On some discontinuous fixed-point mappings in convex metric spaces. Czechoslovak Mathematical Journal. 1993;43(2):319-26.
13. Ding XP. Iteration processes for nonlinear mappings in convex metric spaces. Journal of mathematical analysis and applications. 1988 May 15;132(1):114-22.
14. Georgescu F. Iterated function systems consisting of generalized convex contractions in the framework of complete strong b-metric spaces. Annals of West University of Timisoara-Mathematics and Computer Science. 2017;55(2):119-42.
15. Sawangsup K, Sintunavarat W, Cho YJ. Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces. Journal of Fixed-Point Theory and Applications. 2020 Mar; 22:1-4.
16. Karaca N, Yildirim I. Fixed point theorem for Reich contraction mapping in convex b-metric spaces. Aligarh Bull. Math. 2021; 40:29-39.
17. Sharma DK. δ-convex Structure on Rectangular Metric Spaces concerning Kannan-type Contraction and Reich-type Contraction. 2022;29(4):293-306.
18. Dragomir, S.S. & Toader, G.H.: Some Inequalities for m-Convex Functions. Studia Univ. Babes-Bolyai, Math.1993;38(1):2128.
19. Amirbostaghi G, Asadi M, Mardanbeigi MR. m-convex structure on b-metric spaces. Filomat. 2021;35(14):4765-76.
20. Chen L, Li C, Kaczmarek R, Zhao Y. Several fixed-point theorems in convex b-metric spaces and applications. Mathematics. 2020 Feb 14;8(2):242.

