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Analysis of Simply Supported and Clamped-Clamped Beam Resting on a Pasternak Foundation Subjected to Moving Load with a Damping Term

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Abstract

A comparative analysis of simply supported and clamped-clamped Bernoulli-Euler beams resting on a Pasternak foundation under the action of a concentrated moving load with a damping term, when the inertia effect of the moving load is considered in its analysis, was investigated in this study. The governing equation was solved by representing the Dirac delta function through a Fourier cosine series and applying a combination of analytical techniques, including the generalized finite integral transform, Struble's asymptotic method, and the Laplace transform. The resulting solutions were then evaluated under both simply supported and clamped-clamped boundary conditions to determine the beam's dynamic response. In both cases, the plotted results revealed that the amplitude of vibration decreases progressively as the damping coefficient, shear modulus, foundation stiffness, and axial load parameters increase. However, the deflection profiles of the simply supported beam are generally higher for the moving force problem. In addition, the damping term had a far higher effect on the beam's deflection for the moving force case when compared with other parameters. Also, the moving force problem is not a safe approximation for the moving mass problem. These findings are in agreement with those in literature. Furthermore, the clamped-clamped condition had deflections to the far right, while that of the simply supported condition was typically at the midpoint of the graph. Thus, the beam's safety and longevity are ensured when the values of each parameter increase, and the introduction of the damping term will ensure safety over other parameters.

Keywords: Beam, clamped-clamped support, damping, simply supported, Pasternak foundation

INTRODUCTION

The study of dynamic responses of structural members modelled as beams, such as railway tracks, bridges, runways, and roadways resting on an elastic foundation, subjected to moving loads such as trains and vehicles, is important by virtue of the effect these loads have on them. Usually, the movement of these loads on them induces vibrations, which causes corrosion between contacting elements, impairs the function and life of the structure or its components, and their final failures. One of the earliest researchers to undertake studies on the devastating effects of these vibrations on structural members traversed by loads is Stokes [1]. Over the years, the moving load problem has continued to attract researchers in the fields of applied mathematics, mechanical engineering, and applied physics to provide insights into the design of dynamical structures that will ensure their safety, longevity, and efficient performance. One of the key aspects in the study of the dynamic system is the boundary conditions applied in studying the behavior of the beam. There are the classical boundary conditions and the non-classical

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conditions. Some researchers considered beam supports on non-classical conditions [2–4], while others applied the classical boundary conditions in their study. The classical boundary conditions include simply supported or pin-pin, clamped-clamped or fixed-fixed end, and the clamped-free or cantilever boundary conditions.

One of the earliest studies to employ the simply supported boundary condition was conducted by Krylov, who examined the dynamic behavior of a beam subjected to a constant load moving at uniform velocity, using the eigenfunction expansion method to derive the solution [5]. Subsequently, Timoshenko analyzed a finite, simply supported beam through energy-based techniques, obtaining the corresponding solutions in a series form [6]. Later, Ayre *et al.* explored how the ratio between the moving load's weight and that of the beam influences the dynamic response under a continuously moving mass [7]. Much later, both gravity and inertia effects of the distributed loads were taken into consideration by Oni and Ayankop-Andi, who investigated the problem of a simply supported non-uniform Rayleigh beam under travelling distributed loads [8], while Ogunbamike investigated the dynamic response of the Timoshenko beam resting on an elastic foundation subjected to harmonic moving load using modal analysis (MA) [9]. Several other researchers also examined beams with simply supported boundaries [10–15]. In contrast, studies focusing exclusively on the clamped-clamped boundary condition are comparatively fewer. One such work is that of Lee, who developed a numerical solution using integration schemes based on the Runge-Kutta method to evaluate the dynamic response of a clamped-clamped beam subjected to a moving mass [16]. More recently, Awodola *et al*. analyzed the dynamic behavior of a non-uniform Bernoulli-Euler beam with clamped-clamped ends, supported on a Pasternak elastic foundation, under variable distributed moving loads [17]. Furthermore, several researchers undertook studies where they considered all three classical boundary conditions stated earlier [18–21]. So far, studies where both the simply supported and the clamped-clamped conditions were considered have not been presented.

Interestingly, there were researchers who considered both the simply support and the clamped-clamped conditions in a single study. They include Mirzabeigy and Madolia, who studied the large amplitude free vibration of beams resting on a variable elastic foundation [22]. They studied the effects of variable elastic foundation, amplitude of vibration, and axial load on the nonlinear frequency of beams with simply supported and fully clamped boundary conditions. Still considering a free vibration analysis but using a variable foundation, Kumar also conducted free vibration analysis on an axially functionally graded Euler-Bernoulli beam resting on a variable Pasternak foundation [23]. Two types of boundary conditions, namely, clamped and simply supported, were used in the analysis. Similarly, using a variable foundation, Olotu *et al.* examined the effect of a variable Winkler foundation on the natural frequencies of a prestressed non-uniform Rayleigh beam [24]. The clamped-clamped and simply supported boundary conditions were considered to illustrate the accuracy and efficiency of this method. Although the objective of the researchers who considered cases of simply supported and clamped-clamped support conditions was not to compare their effects on the beam's behavior, it should, however, be noted that they yielded different beam behaviors. There is therefore the need to objectively undertake studies where these alternate boundary conditions are analytically compared with a view to providing insight into the preferred case in the design and performance of beams used in engineering constructions.

It is worth reporting that some researchers objectively compared the effect of simply supported and clamped-clamped boundary conditions on the transverse displacement of beams and got useful results. For instance, Mehmood *et al.* carried out a vibration analysis of beams subjected to moving loads using the finite element method (FEM) [25]. Their findings revealed that beams with pinned-pinned supports exhibited the largest dynamic displacements. Under this boundary condition, the dynamic magnification factors were consistently higher than those observed for clamped-clamped and clamped-pinned configurations at both low and high load velocities. Generally, the clamped-clamped beams displayed the lowest dynamic magnification values, except within the intermediate speed range. Similarly, Ahmadi and Abedi studied the vibrational analysis of beams and found that the beam with clamped-clamped ends had a lower transient deflection and the clamped-clamped beam had a higher critical velocity with respect to the beam with simple-simple end conditions [26]. Regarding the frequency behavior of beams, Usman *et al.* performed a free vibration analysis on non-uniform Euler-Bernoulli double beams resting on a Winkler foundation, considering both simply supported and fixed-fixed boundary conditions [27]. Their results showed that beams with fixed-fixed supports exhibited higher natural frequencies compared to those with simply supported ends. Furthermore, the study revealed that increasing the degree of cross-sectional non-uniformity led to a reduction in the natural frequencies, indicating a sensitivity of the vibrational response to geometric variation. In a related study, Chen *et al.* investigated the resistance behaviors of a clamped beam using a membrane approach [28]. The study indicated that the simply supported beam has a significantly lower load-carrying capacity compared with the clamped beam, and with respect to deflection, it is clear that the peak deflections of the reference beam increase greatly in comparison with the clamped beam. In a recent study on beam frequency, De Rosa *et al.* found that the change in natural frequencies depends largely on the boundary conditions and the length/depth ratios of the beam [29]. Boundary conditions simply supported beams have lower fundamental frequencies than clamped beams. Others are Tagarielli and Fleck [30] and Kumar *et al.* [31].

From the studies reported so far, authors made use of the Winkler or two-parameter foundation, in particular, Pasternak, and got useful results for the dynamical case they considered. However, most of them did not consider damping. Furthermore, from the studies reviewed and to the authors' knowledge in available literature, the case where a comparison is made of the simply supported and clamped-clamped condition for a uniform Bernoulli-Euler beam resting on a Pasternak foundation with a damping term, where the inertia effect of the moving load is considered in its analysis, has not been considered. This study addresses this gap in literature.

FORMULATION OF THE PROBLEM

The governing equation for a uniform Bernoulli Euler beam resting on a constant Pasternak foundation, subjected to a concentrated moving load with a damping term is given as:

$$EI\frac{\partial^4 W(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 W(x,t)}{\partial t^2} + \alpha\frac{\partial W(x,t)}{\partial t} - N\frac{\partial^2 W(x,t)}{\partial x^2} + F_p(x,t) = P(x,t) \quad (1)$$

Where, x is the spatial coordinate, t is the time, $W(x,t)$is the transverse displacement, E is the Young's modulus, I is the moment of inertia, EI is the flexural rigidity of the structure, $\bar{m}$ is the mass per unit length of the beam, α is the damping term coefficient, N is the axial force, $P(x,t)$ is the transverse concentrated load and $F_p(x,t)$ is the foundation reaction. The boundary conditions of the structure are arbitrary, while the initial condition is given as:

$$W(x,0) = 0 = \frac{\partial W(x,t)}{\partial t} \quad (2)$$

The foundation reaction was given by Fryba as [32]:

$$F_p(x,t) = KW(x,t) - G\frac{\partial^2 W(x,t)}{\partial x^2} \quad (3)$$

Where, K is the foundation modulus and G is the shear modulus.

If the inertia effect of the moving load is considered, the load $P(x,t)$ as given by Fryba takes the form [32]:

$$P(x,t) = P_f(x,t)\left[1 - \frac{1}{g}\frac{d^2 W(x,t)}{dt^2}\right] \quad (4)$$

Where, the continuous moving force, $\boldsymbol{P_f(x,t)}$ acting on the beam model is given as:

$$P_f(x,t) = mg\delta(x - ct) \quad (5)$$

Where, m and c are the mass and the speed of the moving load, respectively. g is the acceleration due to gravity, $mg\delta(x-ct),$ is the continuos moving force acting on the beam and $\frac{d^2}{dt^2}$is a convective acceleration defined as:

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2} \tag{6}$$

when the operator, $\frac{d^2}{dt^2}$acts as the transverse deflection $W(x,t)$ of the beam, the first term in the right-hand side of Eq. (6), measures the effect of the acceleration on the deflection, the second term measures the effect of complementary acceleration (Coriolis force), and the third term measures the effect of the path curvature (centripetal force).

Using Eqs. (3)–(6) in Eq. (1), after simplifications, we obtain:

$$\frac{EI}{\bar{m}}\frac{\partial^4 W(x,t)}{\partial x^4} + \frac{\partial^2 W(x,t)}{\partial t^2} - \frac{N}{\bar{m}}\frac{\partial^2 W(x,t)}{\partial x^2} - \frac{G}{\bar{m}}\frac{\partial^2 W(x,t)}{\partial x^2} + \frac{\alpha}{\bar{m}}\frac{\partial W(x,t)}{\partial t} + \frac{k}{\bar{m}}W(x,t) +$$
$$\frac{m}{\bar{m}}\delta(x-ct)\left\{\frac{\partial^2 W(x,t)}{\partial t^2} + 2c\frac{\partial^2 W(x,t)}{\partial x \partial t} + c^2\frac{\partial^2 W(x,t)}{\partial x^2}\right\} = \frac{mg}{\bar{m}}\delta(x-ct) \tag{7}$$

Where, $\delta(x-ct)$ represents the Dirac delta function defined as:

$$\delta(x-ct) = \begin{cases} 0, x \neq ct \\ \infty, x = ct \end{cases} \tag{8}$$

Eq. (7) is the simplified governing equation for a uniform Bernoulli Euler beam resting on a constant Pasternak foundation subjected to a concentrated moving load with a damping term, when the inertia effect of the moving load is put into consideration.

METHOD OF SOLUTION

To solve Eq. (7) with the associated boundary conditions, a general approach is resorted to, which involves expressing the Dirac delta function as a Fourier cosine series and then reducing the modified form of the fourth (4th) order PDE above using the generalized finite integral transform. The generalized finite integral transform is defined by:

$$W(m,t) = \int_0^L W(x,t)U_m(x)dx \tag{9}$$

with the inverse transform as:

$$W(x,t) = \sum_{m=1}^{\infty}\frac{\bar{m}}{W_m}W(m,t)U_m(x) \tag{10}$$

Where,

$$W_m = \int_0^L \bar{m}\, U_m^2(x)dx \tag{11}$$

And $U_m(x)$ is any function chosen such that the pertinent boundary condition is satisfied. Thus, the m^{th} normal mode of vibration of a uniform beam is given as:

$$U_m(x) = \mathrm{Sin}\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sin\frac{\lambda_m x}{L} + C_m Cos\frac{\lambda_m x}{L} \tag{12}$$

is chosen as a suitable kernel of the integral transform Eq. (9), where $\lambda_m$is the node frequency, A_m, B_m and C_m are constants which are obtained by substituting the boundary condition into Eq. (12).

By applying the generalized finite integral transform Eq. (9) on Eq. (7), we obtain:

$$\frac{EI}{\bar{m}}\int_o^L \frac{\partial^4 W(x,t)}{\partial x^4}U_m(x)dx + \int_o^L \frac{\partial^2 W(x,t)}{\partial t^2}U_m(x)dx - \frac{N}{\bar{m}}\int_o^L \frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx - \frac{G}{\bar{m}}\int_o^L \frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx + \frac{\alpha}{\bar{m}}\int_o^L \frac{\partial W(x,t)}{\partial t}U_m(x)dx + \frac{k}{\bar{m}}\int_o^L W(x,t)U_m(x)dx - \frac{M}{\bar{m}}\int_o^L \delta(x-ct)\frac{\partial^2 W(x,t)}{\partial t^2}U_m(x)dx - \frac{2Mc}{\bar{m}}\int_o^L \delta(x-ct)\frac{\partial^2 W(x,t)}{\partial x \partial t}U_m(x)dx + \frac{Mc^2}{\bar{m}}\int_o^L \delta(x-ct)\frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx = \frac{Mg}{\bar{m}}\int_o^l \delta(x-ct)U_m(x)dx \tag{13}$$

we use the property of the Dirac delta function as an even function to express the above term in Fourier cosine series as:

$$\delta(x-ct) = \frac{1}{L} + \frac{2}{L}\sum_{n-1}^{\infty}\cos\frac{n\pi ct}{L}\cos\frac{n\pi x}{L} \tag{14}$$

Using Eq. (14) on (13) and after some simplification and rearrangements, Eq. (13) becomes:

$$W_{tt}(m,t) + \frac{\alpha}{\bar{m}}W_t(m,t) + \left[\frac{EI}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^4 + \frac{N}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^2 + \frac{G}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^2 + \frac{K}{\bar{m}}\right]W(m,t) + \frac{EI}{\bar{m}}E_1(x) - \left(\frac{N+G}{\bar{m}}\right)E_2(x) - \frac{M}{\bar{m}L}U_1(m,t)W_{tt}(m,t) - \frac{M}{\bar{m}L}P_1(k,m,t)W_{tt}(k,t) - \frac{2Mc}{\bar{m}L}U_2(m,t)W_t(m,t) - \frac{2Mc}{\bar{m}L}P_2(k,m,t)W_t(k,t) - \frac{Mc^2}{\bar{m}L}U_3(m,t)W(m,t) - \frac{Mc^2}{\bar{m}L}P_3(k,m,t) = \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_{mct}}{L} + A_m\cos\frac{\lambda_m ct}{L} + B_m\sinh\frac{\lambda_m ct}{L} + C_m\cosh\frac{\lambda_m ct}{L}\right] \tag{15}$$

Where,

$$U_1(m,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[H_a(m,n) + 2H_c(m,n)\cos\frac{n\pi ct}{L}\right] \tag{16}$$

$$U_2(m,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[H_e(m,n) + 2H_g(m,n)\cos\frac{n\pi ct}{L}\right] \tag{17}$$

$$U_3(m,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[H_i(m,n) + 2H_k(m,n)\cos\frac{n\pi ct}{L}\right] \tag{18}$$

$$P_1(k,m,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[H_b(k,m,n) + 2H_d(k,m,n)\cos\frac{n\pi ct}{L}\right] \tag{19}$$

$$P_2(k,m,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[H_f(k,m,n) + 2H_L(k,m,n)\cos\frac{n\pi ct}{L}\right] \tag{20}$$

$$P_3(k,m,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[H_j(k,m,n) + 2H_L(k,m,n)\cos\frac{n\pi ct}{L}\right] \tag{21}$$

Rearranging Eq. (15):

$$W_{tt}(m,t) + \frac{\alpha}{\bar{m}}W_t(m,t) + \left[\frac{EI}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^4 + \left(\frac{N+G}{\bar{m}}\right)\left(\frac{\lambda_m}{L}\right)^4 + \frac{K}{\bar{m}}\right]W(m,t) + \frac{EI}{\bar{m}}E_1(x) - \left(\frac{N+G}{\bar{m}}\right)E_2(x) - \varepsilon_0[U_1(m,t)W_{tt}(m,t) + P_1(k,m,t)W_{tt}(k,t) + 2cU_2(m,t)W_t(m,t) + 2cP_2(k,m,t)W_t(k,t) + c^2U_3(m,t)W(m,t) + c^2P_3(k,m,t)W(k,t)] = \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m\cos\frac{\lambda_m ct}{L} + B_m\sinh\frac{\lambda_m ct}{L} + C_m\cosh\frac{\lambda_m ct}{L}\right] \tag{22}$$

Where,

$$E_1(x) = E_1(x) = \left[\frac{\partial^3}{\partial x^3}W(x,t)U_m(x) - \frac{\partial^2}{\partial x^2}W(x,t)\frac{dU_m(x)}{dx} + \frac{\partial}{\partial x}w(x,t)\frac{d^2}{dx^2}U_m(x) - W(x,t)\frac{d^3}{dx^3}U_m(x)\right]_0^L \tag{23}$$

$$E_2(x) = \left[\frac{\partial}{\partial x}W(x,t)U_m(x) - W(x,t)\frac{d}{dx}U_m(x)\right]_0^L \tag{24}$$

$$\varepsilon_0 = \frac{M}{\bar{m}L} \tag{25}$$

Rearranging Eq. (22), it becomes:

$$W_{tt}(m,t) + A_1 W_t(m,t) + A_2 W(m,t) + A_3(x) - \varepsilon_0[U_1(m,t)W_{tt}(m,t) + 2cU_2(m,t)W_t(m,t) + c^2 U_3(m,t)w(m,t)] - \varepsilon_0[P_1(k,m,t)W_{tt}(k,t) + 2cP_2(k,m,t)W_t(k,t) + c^2 P_3(k,m,t)W(k,t)] = \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m \cos\frac{\lambda_m ct}{L} + B_m \sinh\frac{\lambda_m ct}{L} + C_m \cosh\frac{\lambda_m ct}{L}\right] \tag{26}$$

Where,

$$A_1 = \frac{\alpha}{\bar{m}} \tag{27}$$

$$A_2 = \frac{EI}{\bar{m}}\left(\frac{\lambda_m}{L}\right)^4 + \left(\frac{N+G}{\bar{m}}\right)\left(\frac{\lambda_m}{L}\right)^4 + \frac{k}{\bar{m}} \tag{28}$$

$$A_3(x) = \left[\frac{EI}{\bar{m}} E_1(x) - \left(\frac{N+G}{\bar{m}}\right) E_2(x)\right]_0^L \tag{29}$$

Eq. (26) is the transformed equation governing the problem. This non-homogeneous 2nd order ODE holds for all variants of the classical Boundary Conditions.

Specifically, considering the simply supported boundary condition, $W(0,t) = 0 = W''(0,t)$ and $W(L,t) = 0 = W''(L,t)$Likewise, $U_m(0) = U_1''(0) = 0$ and $U_m(L) = U''(L) = 0$

Slotting these into Eqs. (23) and (24), we obtain the following:

$$E_1(x)\Big|_0^L = 0 \tag{30}$$

And

$$E_2(x)\Big|_0^L = 0 \tag{31}$$

$$\Rightarrow A_3(x)\Big|_0^L = 0 \tag{32}$$

In what follows, two special cases of Eq. (**26**) are considered.

Uniform Bernoulli-Euler Beam Under a Concentrated Moving Force

An approximate form of the governing differential equation for a uniform Bernoulli-Euler beam resting on an elastic Pasternak foundation and subjected to a concentrated moving force can be derived by neglecting the inertia effect of the moving mass. Under this assumption, $A_3(x) = 0$ and ε_0=0, causing Eq. (26) to simplify accordingly:

$$W_{tt}(m,t) + A_1 W_t(m,t) + A_2 W(m,t) = \frac{Mg}{\bar{m}}\left[\sin\frac{\lambda_m ct}{L} + A_m \cos\frac{\lambda_m ct}{L} + B33333_m \sinh\frac{\lambda_m ct}{L} + \cosh\frac{\lambda_m ct}{L}\right] \tag{33}$$

In order to solve Eq. (33), the method of Laplace transform with the Convolution theorem and substitution of Eq. (2) is done. The following are some of the relations employed:

$$F(s) = \int_0^\infty e^{-st} f(t)dt\,; \;\; s > 0 \tag{34}$$

Where, $'s'$ is the Laplace parameter in conjunction with Eq. (2)

The following representations were adopted to reduce the problem to finding the Laplace inversion:

$$\left.\begin{aligned} F_f(s) &= \frac{\theta m}{s^2+\theta_m^2} + \frac{sAm}{s^2+\theta_m^2} + \frac{\theta m Bm}{s^2-\theta_m^2} + \frac{s\theta m}{s^2-\theta_m^2} \\ G_{1f}(s) &= \frac{1}{s+r_2},\ \ G_{2f}(s) = \frac{1}{s+r_1} \end{aligned}\right\} \quad (35)$$

After some substitutions, evaluations, simplifications, and substituting into Eq. (10), we obtained:

$$W(x,t) = \sum_{m=1}^{a} Q \left\{ \begin{array}{l} \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2+r_2{}^2}\left[e^{-r_2 t} - cos\theta_m t + \frac{r_2}{\theta_m} sin\theta_m t\right] \\ -\frac{\theta_m}{\theta_m{}^2+r_1{}^2}\left[e^{-r_1 t} - cos\theta_m t + \frac{r_1}{\theta_m} sin\theta_m t\right] \end{array} \right] \\ +A_m \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2+r_2{}^2}\left[sin\theta_m t + \frac{r_2}{\theta_m}(cos\theta_m t - e^{-r_2 t})\right] \\ -\frac{\theta_m}{\theta_m{}^2+r_1{}^2}\left[sin\theta_m t + \frac{r_1}{\theta_m}(cos\theta_m t - e^{-r_1 t})\right] \end{array} \right] \\ +B_m \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2-r_2{}^2}\left[cosh\theta_m t - e^{-r_2 t} - \frac{r_2}{\theta_m} sinh\theta_m t\right] \\ -\frac{\theta_m}{\theta_m{}^2-r_1{}^2}\left[cosh\theta_m t - e^{-r_1 t} - \frac{r_1}{\theta_m} sinh\theta_m t\right] \end{array} \right] \\ +C_m \left[\begin{array}{l} \frac{\theta_m}{\theta_m{}^2-r_2{}^2}\left[sinh\theta_m t - \frac{r_2}{\theta_m}(e^{-r_2 t} - cosh\theta_m t)\right] - \\ \frac{\theta_m}{\theta_m{}^2-r_1{}^2}\left[sinh\theta_m t - \frac{r_1}{\theta_m}(e^{-r_1 t} - cosh\theta_m t)\right] \end{array} \right] \end{array} \right\} U_m(x) \quad (36)$$

Eq. (**36**) represents the transverse displacement response of the beam under the action of a moving concentrated force.

Uniform Bernoulli-Euler Beam Traversed by Concentrated Moving Mass

When the mass of the moving load is considered, the inertia effect of the moving mass is not negligible. Thus, $\varepsilon_o \neq 0$ and the solution to the entire Eq. (26) is required. Setting$A_3(x) = 0$, retaining the inertia term ε_o in Eq. (26) and applying strubble asymptotic method with further arrangement, Eq. (26) becomes:

$$\begin{aligned} &W_{tt}(m,t) + A_1 W_t(m,t) + A_2 W(m,t) - \varepsilon_o[U_1(m,t)W_{tt}(m,t) + \\ &\quad 2CU_2(m,t)W_t(m,t) + C^2U_3(m,t)W(m,t)] - \varepsilon_o[P_1(k,m,t)W_{tt}(k,t) + \\ &\quad 2CP_2(k,m,t)W_t(k,t) + C^2P_3(k,m,t)W(k,t)] \\ &\quad = \frac{Mg}{m}[Sin\theta_m t + A_m cos\theta_m t + B_m sinh\theta_m t + C_m cosh\theta_m t] \end{aligned} \quad (37)$$

Where,

$$\varepsilon_o = \frac{M}{\bar{m}L} \quad (38)$$

$$\theta_m = \frac{\lambda_m c}{L} \quad (39)$$

Let us consider a parameter $\in < \mathbf{1}$ for any arbitrary mass ratio $\boldsymbol{\varepsilon_o}$, defined as:

$$\in = \frac{\boldsymbol{\varepsilon_o}}{\mathbf{1} + \boldsymbol{\varepsilon_o}} \quad (40)$$

$$\varepsilon_o = \in (1 - \in)^{-1} \quad (41)$$

From the theory of Binomial expansion and truncating after the second term, one obtains:

$$\varepsilon_o = \in + 0(\in^2) \quad (42)$$

Applying Eqs. (40)–(42) and after some evaluation, substitutions, simplifications, and rearrangements, we obtain:

$$\begin{aligned} &-2\dot{\emptyset}(m,t)\,\alpha_{mf}\sin\left[\alpha_{mf}t - \Omega(m,t)\right]\ 2\emptyset(m,t)\,\alpha_{mf}t\dot{\Omega}(m,t)Cos[\,\alpha_{mf}t - \Omega(m,t)] - \\ &\quad \emptyset(m,t)\,\alpha_{mf}{}^2\cos\left[\alpha_{mf}t - \Omega(m,t)\right] + \in \dot{W}_1(m,t) - \in(A_1 + \\ &\quad R_2(t))\,\emptyset(m,t)\,\alpha_{mf}\sin\left[\alpha_{mf}t - \Omega(m,t)\right] + \in(A_2 + R_3(t))\,[\,\emptyset(m,t)\cos[\,\alpha_{mf}t - \end{aligned}$$

$$\Omega(m,t)]+\in \{P_1[\emptyset(k,t)\,\alpha_{mf}{}^2\cos[\alpha_{mf}t-\Omega(k,t)]+2cP_2\,[\emptyset(k,t)\,\alpha_{mf}\sin[\alpha_{mf}t-\Omega(k,t)]-C^2P_3[\emptyset(k,t)\cos[\alpha_{mf}t-\Omega(k,t)]]\}=0 \quad (43)$$

In order to obtain the modified frequency, we extract only the variation part of Eq. (43) that describes the behavior of $\emptyset(m,t)$ and $\Omega(m,t)$ during the motion of mass.

After some evaluation, simplification and substitution, we obtain:

$$W(m,t)=Ae^{\frac{-\in(A_1+R_2)}{2}t}\cos\left[\alpha_{mf}t-\{\frac{\alpha_{mf}{}^2-\in(A_2-\frac{C^2U_3(m,t)}{\in})}{2\,\alpha_{mf}}t+C_m\}\right] \quad (44)$$

$$W(m,t)=Ae^{\frac{-\in(A_1+R_2)}{2}t}cos\,[\alpha_{mm}t+Cm\,] \quad (45)$$

$$\text{Where } \alpha_{mm}=\alpha_{mf}-\left(\frac{\alpha_{mf}{}^2+C^2U_3(m,t)-\in A_2}{2\,\alpha_{mf}}\right) \quad (46)$$

Eq. (46) is called the modified natural frequency.

Thus, to solve the non-homogeneous equation in $Eq.$ (26), the differential operator which acts on $W(m,t)$ is replaced by the modified natural frequency α_{mm}. Hence, we have:

$$W_{tt}(m,t)+A_1W_t(m,t)+\alpha_{mm}W(m,t)=\frac{Mg}{\bar{m}}[Sin\theta_mt+A_m\,Cos\theta_mt+BSinh\theta_mt+CmCosh\theta_mt] \quad (47)$$

To solve Eq. (47), the method of Laplace transformation and convolution theorem is resorted to, as in the case of moving force, to obtain:

$$W(x,t)=\sum_{m=1}^{\infty}Q_1\left\{\begin{array}{l}\left[\begin{array}{l}\frac{\theta_m}{\theta_m{}^2+w_2{}^2}[e^{-w_2t}-Cos\theta_mt+\frac{w_2}{\theta_m}Sin\theta_mt]\\-\frac{\theta_m}{\theta_m{}^2+w_1{}^2}[e^{-w_1t}-Cos\theta_mt+\frac{w_1}{\theta_m}Sin\theta_mt]\end{array}\right]\\+A_m\left[\begin{array}{l}\frac{\theta_m}{\theta_m{}^2+w_2{}^2}[Sin\theta_mt+\frac{w_2}{\theta_m}(Cos\theta_mt-e^{-w_2t})]\\-\frac{\theta_m}{\theta_m{}^2+w_2{}^2}[Sin\theta_mt+\frac{w_1}{\theta_m}(Cos\theta_mt\,e^{-w_1t})]\end{array}\right]\\+B_m\left[\begin{array}{l}\frac{\theta_m}{\theta_m{}^2-w_2{}^2}[Cosh\theta_mt-e^{-w_2t}-\frac{w_2}{\theta_m}Sinh\theta_mt\,)]\\-\frac{\theta_m}{\theta_m{}^2-w_1{}^2}[Cosh\theta_mt-e^{-w_1t}-\frac{w_1}{\theta_m}Sinh\theta_mt\,)]\end{array}\right]\\+C_m\left[\begin{array}{l}\frac{\theta_m}{\theta_m{}^2-w_2{}^2}[Sinh\theta_mt-\frac{w_2}{\theta_m}(e^{-w_2t}-Cosh\theta_mt\,)]\\-\frac{\theta_m}{\theta_m{}^2-w_1{}^2}[Sinh\theta_mt-\frac{w_1}{\theta_m}(e^{-w_1t}-Cosh\theta_mt\,)]\end{array}\right]\end{array}\right\}U_m(x) \quad (48)$$

Eq. (48) represents the transverse displacement response of the beam under the action of a moving concentrated mass.

ILLUSTRATIVE EXAMPLES

Application of the Simply Supported Boundary Condition

The simply supported boundary conditions are:

$$W(0,t)=W(L,t)=0,\ \frac{\partial^2w(0,t)}{\partial x^2}=0=\frac{\partial^2w(L,t)}{\partial x^2} \quad (49)$$

Consequently, for the normal modes:

$$U_m(0)=U_m(L)=0,\frac{\partial^2U_m{}^{(0)}}{\partial x^2}=0=\frac{\partial^2U_m{}^{(L)}}{\partial x^2} \quad (50)$$

It is also clear that:

$$U_k(0) = U_k(L) = 0, \quad \frac{\partial^2 U_k{}^{(o)}}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2} \tag{51}$$

Using the above Boundary Condition in Eq. (12), results to:

$A_m = B_m = C_m = 0$
and$A_k = B_k = C_k = 0$

and the frequency equation is obtained as:

$Sin\lambda_m = 0$,
Thus, $\lambda_m = m\pi$
Similarity,
$Sin\lambda_k = 0$ then $\lambda_m = k\pi$

Substituting the $A_m = B_m = C_m = 0$ into Eq. (36)for the moving force problem, one obtains:

$$W(x,t)=\sum_{m=1}^{\infty} Q \begin{bmatrix} \frac{\theta_m}{\theta_m{}^2+r_2{}^2}[e^{-r_2 t}- \cos\theta_m t+\frac{r_2}{\theta_m}\sin\theta_m t] \\ -\frac{\theta_m}{\theta_m{}^2+r_1{}^2}[e^{-r_1 t}- \cos\theta_m t+\frac{r_1}{\theta_m}\sin\theta_m t] \end{bmatrix} U_m(x) \tag{52}$$

Eq. (52) describes the dynamic response of a simply supported uniform Bernoulli-Euler beam for a moving force problem resting on a Pasternak foundation, subjected to a concentrated load with a damping term.

Similarly, substituting the $A_m = B_m = C_m = 0$ into Eq. (48) for the moving mass problem, one obtains:

$$W(x,t) = \sum_{m=1}^{\infty} Q_1 \left[\begin{bmatrix} \frac{\theta_m}{\theta_m{}^2+w_2{}^2}[e^{-w_2 t} - Cos\theta_m t + \frac{w_2}{\theta_m} Sin\theta_m t] \\ -\frac{\theta_m}{\theta_m{}^2+w_1{}^2}[e^{-w_1 t} - Cos\theta_m t + \frac{w_1}{\theta_m} Sin\theta_m t] \end{bmatrix}\right] U_m(x) \tag{53}$$

Again, Eq. (53) describes the dynamic response of a simply supported uniform Bernoulli-Euler beam for a moving mass problem resting on a Pasternak foundation, subjected to a concentrated load with a damping term.

Application of the Camped-Clamped Boundary Condition

The clamped-clamped boundary conditions are:

$$W(0,t) = W(L,t) = 0, \frac{\partial W(0,t)}{\partial x} = 0 = \frac{\partial W(L,t)}{\partial x} \tag{54}$$

For normal modes,

$$U_m(0) = U_m(L) = 0, \frac{\partial U_m(0)}{\partial x} = 0 = \frac{\partial U_m{}^{(L)}}{\partial x} \tag{55}$$

Also $u_m(o) = u_k(L) = 0, \frac{\partial u_k{}^{(o)}}{\partial x} = 0 = \frac{\partial u_k{}^{(L)}}{\partial x}$ (56)

Using the above Boundary Condition in Eq. (12), results in:

For $U_m(x)$, we get $A_m = -C_m$ (57)

and $\frac{\partial U_m(x)}{\partial x}$ gives $B_m = -1$ (58)

Similarly, $U_m(L)$ gives:

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} \tag{59}$$

$\frac{\partial U_m(L)}{\partial x}$ yields the following frequency equation:

$$Cos\lambda_m\ Cosh\lambda_m = 1 \tag{60}$$

With solutions,
$\lambda_1 = 4.73004, \quad \lambda_2 = 7.85320, \ \lambda_3 = 10.99561$
For $m = 1{,}2{,}3$

By substituting Eqs. (57)–(60) into (36) and (48), the expressions corresponding to the moving force and moving mass cases are obtained. These represent the dynamic responses of a clamped-clamped uniform Bernoulli-Euler beam resting on a Pasternak foundation and subjected to a concentrated load incorporating a damping term.

NUMERICAL ANALYSIS AND DISCUSSION OF RESULTS

Numerical analysis for both moving concentrated force and moving concentrated mass problems was carried out for all the parameters considered. In order to obtain the effects of the interacting beam parameters, the analytic results were simulated using MATLAB. The analysis in this work was simulated using MATLAB and illustrated by considering a homogenous beam of modulus of elasticity, $E = 2.02 \times 10^{11} N/m^2$, the moment of inertia $I = 2.87698 \times 10^{-3} m^4$, the beam span $L = 100m$, the mass per unit length of the beam $\bar{m} = 2758.291\ Kg/m$. With respect to the parameters, the values of axial force N was varied between $0N\ and\ 20000000N$, the values of the shear modulus G varies between $0\ N/m^3 and\ 50000000\ N/m^3$, the values of k varies between $0\ N/m^3 and\ 50000000\ N/m^3$ and the values of the damping coefficient α varied between 0 and 6.

DISCUSSION ON BEAMS' DEFLECTION

In Figures 1–18, the deflection profiles of the uniform Bernoulli-Euler beam under the action of a moving concentrated force for various values of damping coefficient α, and shear modulus G, foundation stiffness K, and axial force N, while others were fixed per case for the moving force and moving mass problems were presented. The figure revealed that as α increases and other parameters are fixed, the response amplitude of the uniform Bernoulli-Euler beam decreases. From the results, it is evident that an increase in the values of the four parameters reduced the deflection profile of the beam. In effect, their increase ensured a safe passage of the load and a prolonged beam's life. By comparing the effect of the simply supported and clamped-clamped conditions on the beam's behavior, the study showed that the simply supported condition generally had a higher deflection for both the moving force and the moving mass problems. However, it was observed that the damping coefficient led to more noticeable effects on the deflection profiles of the simply supported beam than the clamped-clamped beam for the moving force than the moving mass problem. In addition, the clamped-clamped condition had a more complex and undulating deflection profile. While the deflection of the simply supported condition concentrated at the mid-point of the graphs, showing the maximum amplitude, the clamped-clamped case deflected more to the right side of the graph. These findings are in line with those of Mehmood, *et al.* [25], Ahmadi (2022) [26], and Chen *et al.* (2024) [28]. Furthermore, the comparison graphs showed that the moving mass problem had a higher deflection profile than the moving force problem for both conditions. This implies that the moving force problem is safer than that of the moving mass. However, it cannot be used as a safe approximation of the moving mass problem.

Comments on Closed Form Solution

The deflections of the uniform Bernoulli-Euler beam may increase beyond bounds, which could cause the failure of the beam or resonance. The beam traversed by a moving concentrated force reaches a state of resonance whenever,

$$\alpha_{mf} = \theta_m = \frac{\lambda_m c}{L}$$

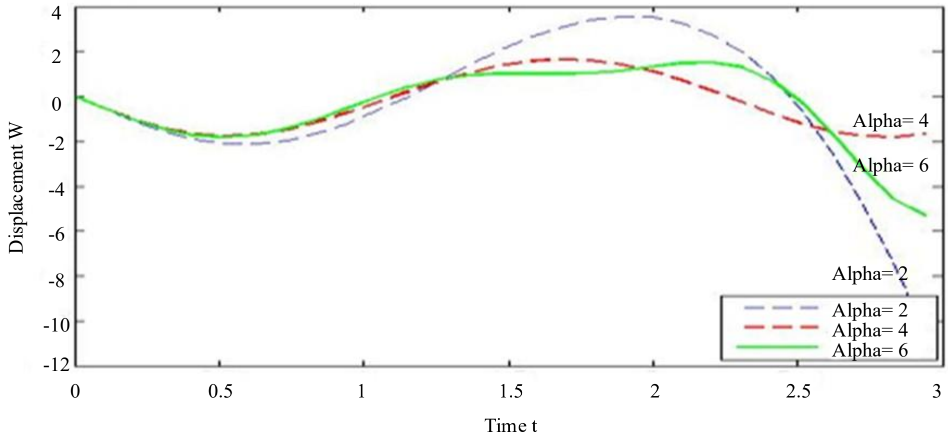

Figure 1. Transverse displacement of a simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for different values of the damping coefficient α, with fixed parameters: shear modulus $G(50000)$, axial force $N(20000)$ and foundation modulus$K(50000)$ under a moving concentrated force.

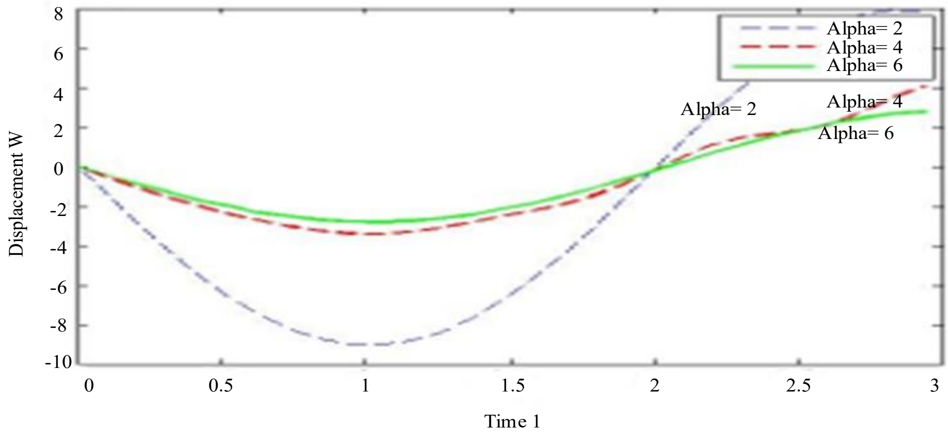

Figure 2. Transverse displacement of a simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for varying values of the damping coefficient α with fixed parameters: shear modulus $G(50000)$, axial force $N(20000)$ and foundation modulus $K(50000)$ under a moving concentrated mass.

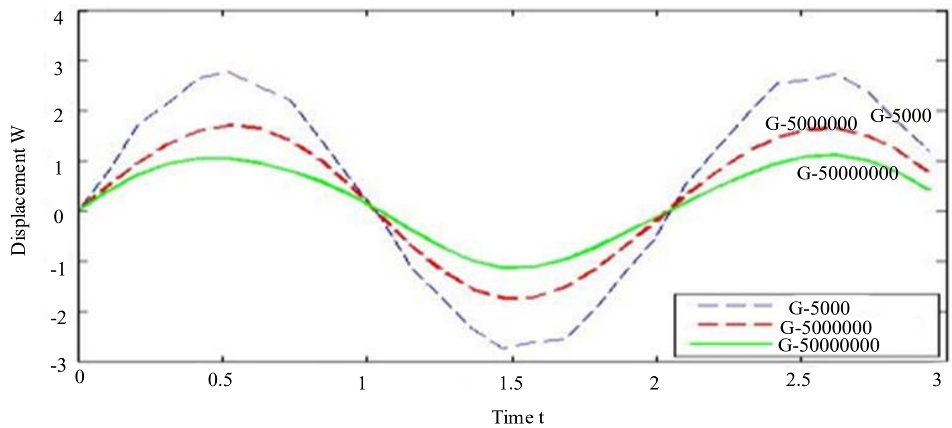

Figure 3. Transverse displacement of the simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for various values of shear modulus G and fixed values of the damping coefficient α(2), axial force $N(20000)$ and foundation modulus $K(50000)$ of a moving concentrated force.

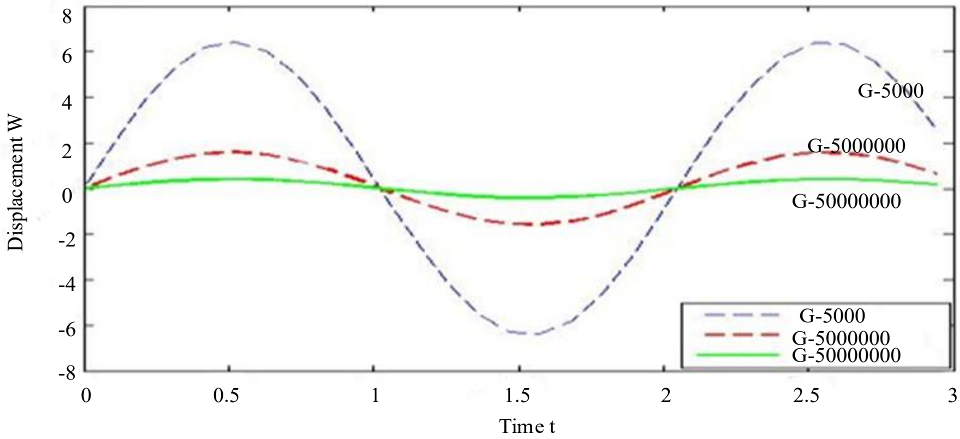

Figure 4. Transverse displacement of the simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for various values of shear modulus G fixed values of the damping coefficient $\alpha(2)$, axial force $N(20000)$ and foundation modulus $K(50000)$ of a moving concentrated mass.

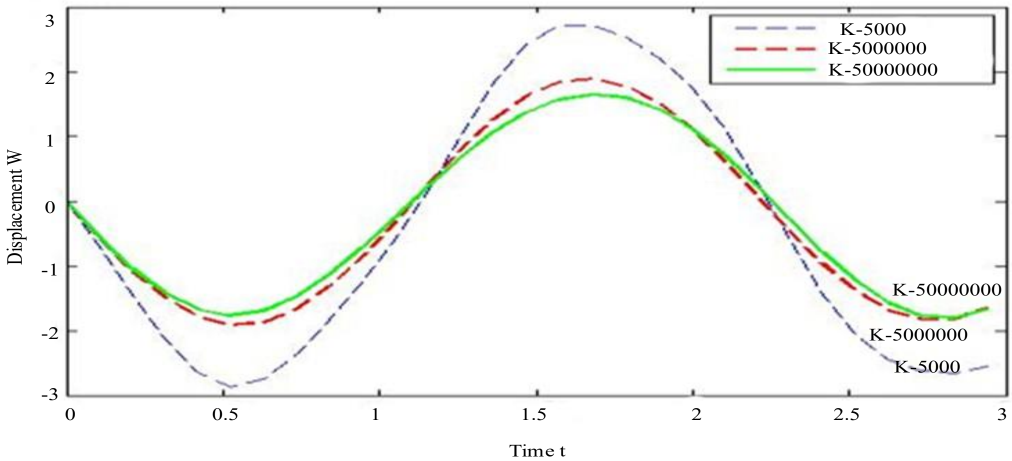

Figure 5. Transverse displacement of the simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for various values of foundation modulus (K) and fixed values of the damping coefficient $\alpha(2)$, axial force $N(20000)$ and shear modulus $G(50000)$ of a moving concentrated force.

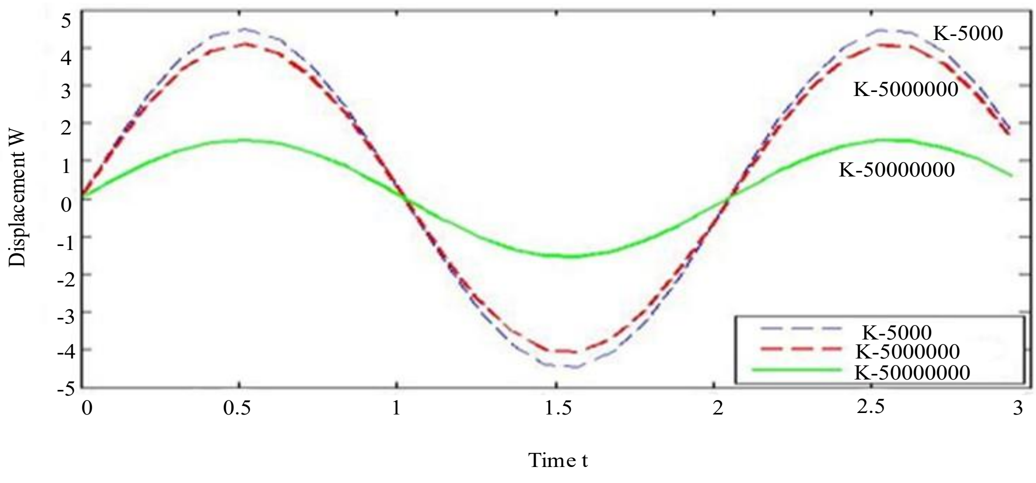

Figure 6. Transverse displacement of the simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for various values of foundation modulus (K) and fixed values of the damping coefficient $\alpha(2)$, axial force $N(20000)$ and shear modulus $G(50000)$ of a moving concentrated mass.

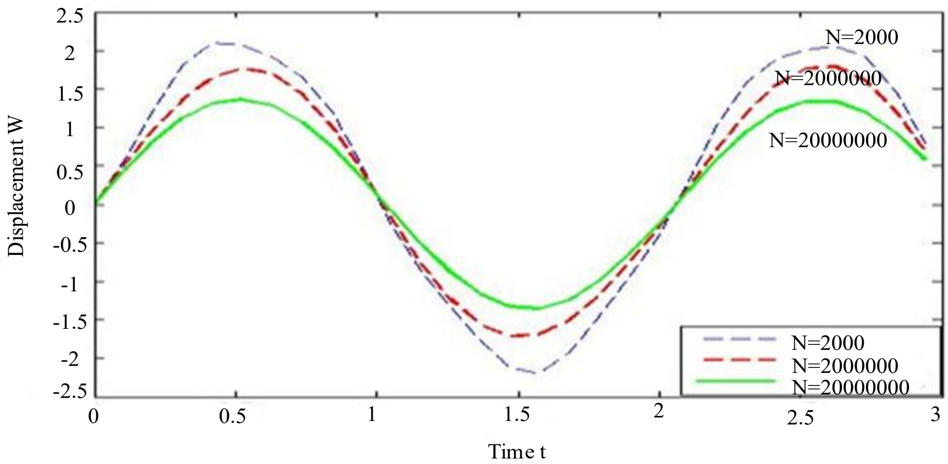

Figure 7. Transverse displacement of the simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for various values of axial force $N$and fixed values of damping coefficient α (2), foundation modulus $K(50000)$ and shear modulus $G(50000)$, of a moving concentrated force.

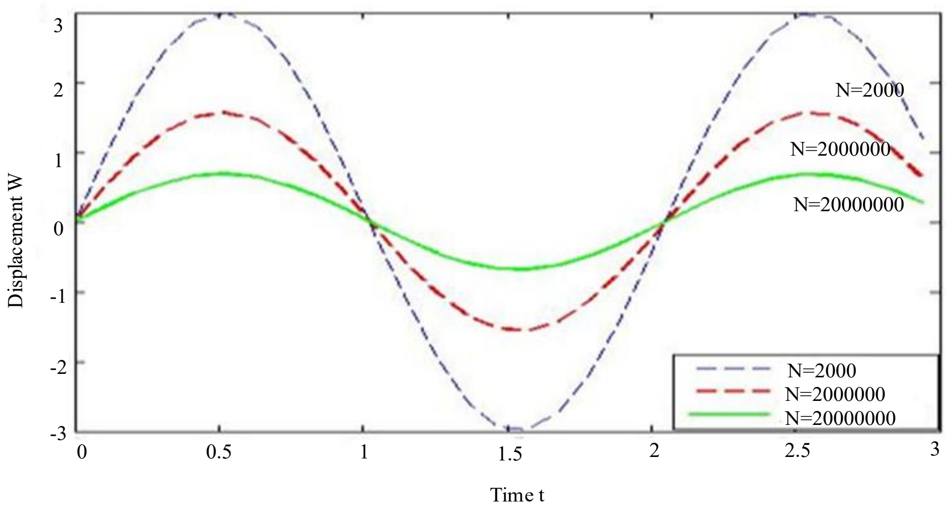

Figure 8. Transverse displacement of the simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation for various values of axial force N and fixed values of damping coefficient α (2), foundation modulus $K(50000)$, and shear modulus $G(50000)$ of a moving concentrated mass.

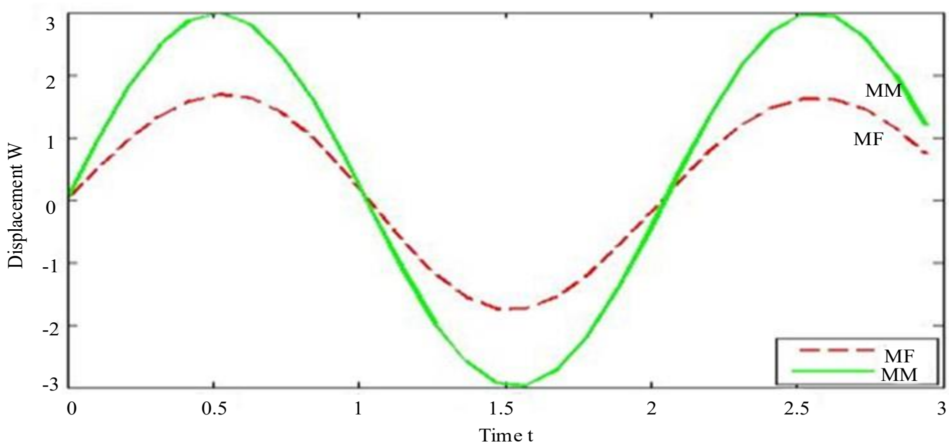

Figure 9. Comparison of the transverse displacement of a simply supported uniform Bernoulli-Euler beam resting on a Pasternak foundation under moving force and moving mass conditions.

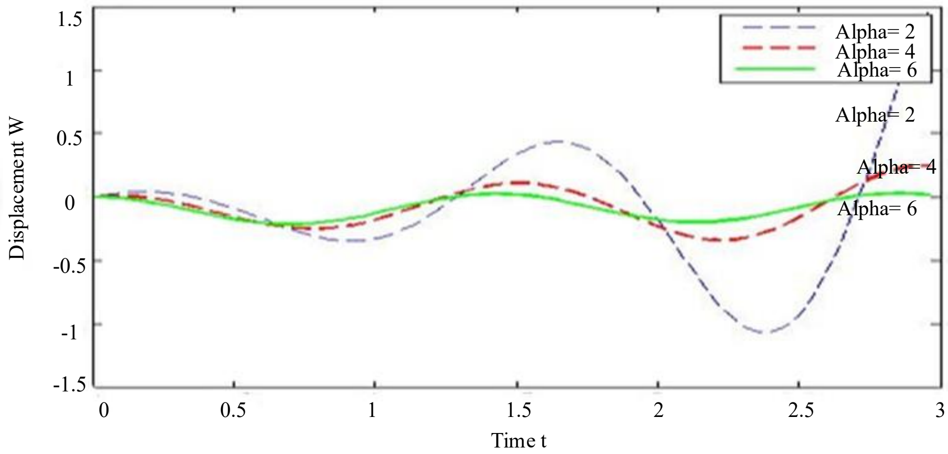

Figure 10. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the damping coefficient α, with fixed parameters: shear modulus *G* (50000), axial force *N* (20000), and foundation modulus *K* (50000), when traversed by a moving concentrated force.

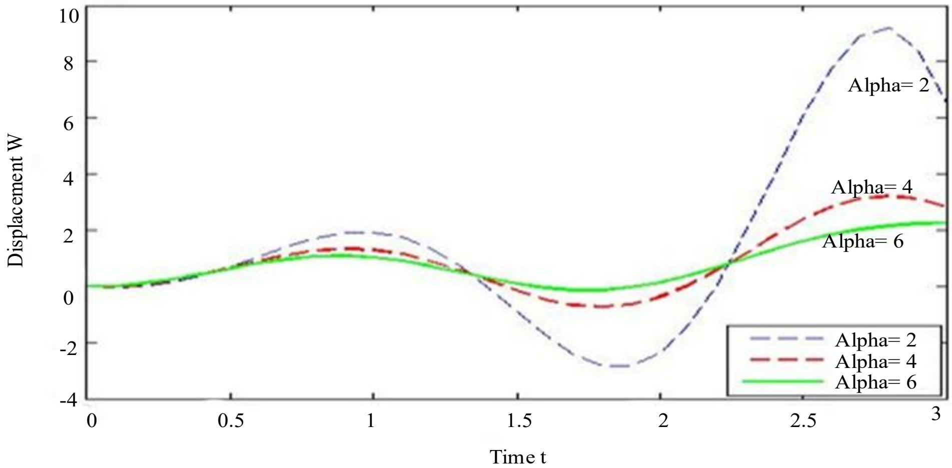

Figure 11. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the damping coefficient α, with fixed parameters: shear modulus *G* (50000), axial force *N* (20000), and foundation modulus *K* (50000), when traversed by a moving concentrated mass.

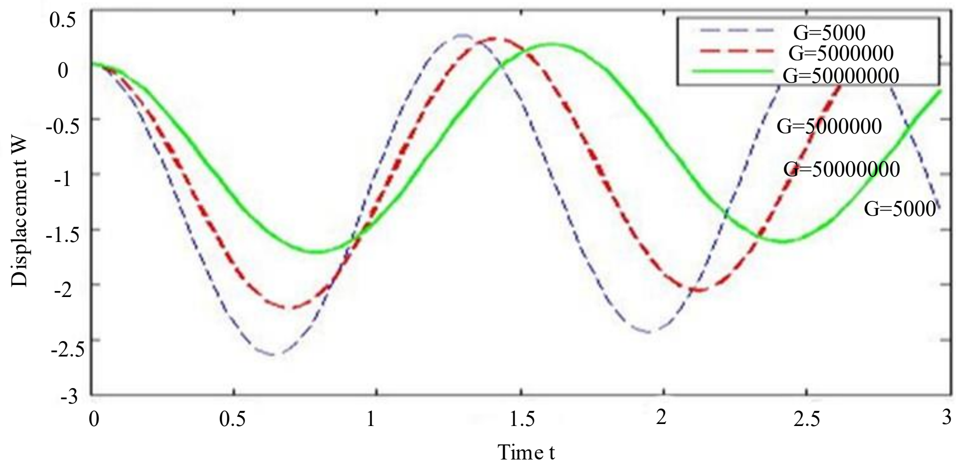

Figure 12. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the shear modulus *G* with fixed parameters: damping coefficient α (2), axial force *N* (20000) and foundation modulus *K* (50000), when traversed by a moving concentrated force.

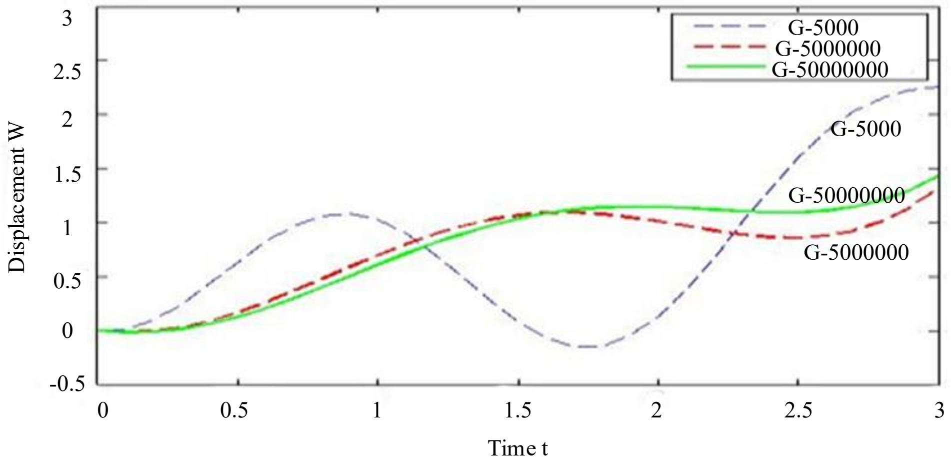

Figure 13. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the shear modulus G with fixed parameters: damping coefficient α (2), axial force N (20000) and foundation modulus K (50000), when traversed by a moving concentrated mass.

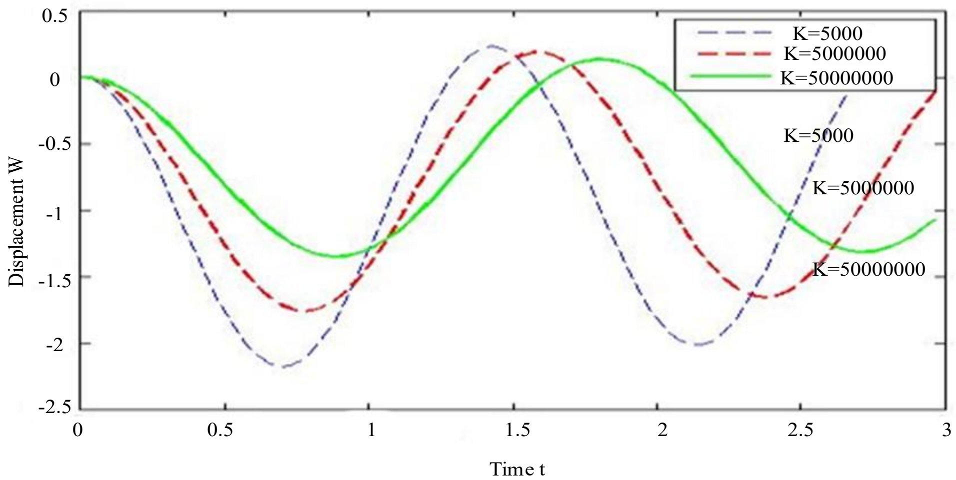

Figure 14. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the foundation modulus (K) with fixed parameters: damping coefficient α (2), axial force N (20000) and shear modulus G (50000), when traversed by a moving concentrated force.

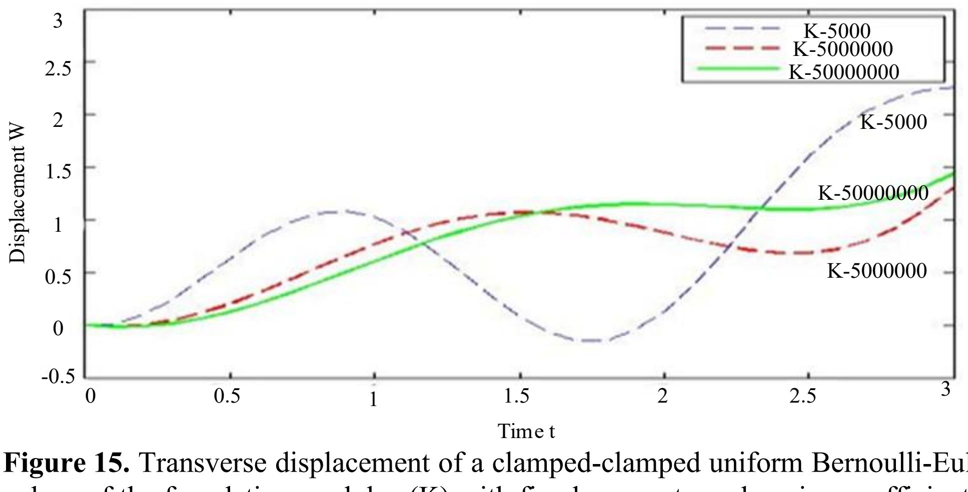

Figure 15. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the foundation modulus (K) with fixed parameters: damping coefficient α (2), axial force N (20000) and shear modulus G (50000), when traversed by a moving concentrated mass.

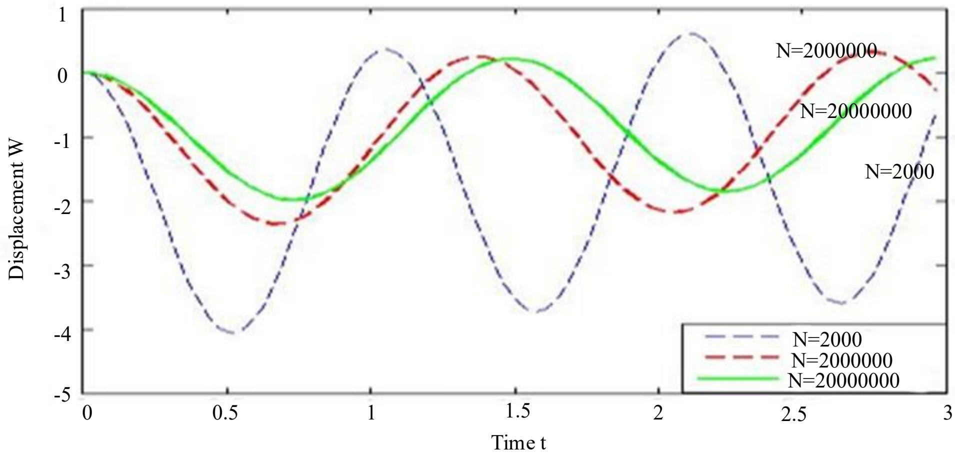

Figure 16. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the axial force *N* with fixed parameters: damping coefficient α (2), foundation modulus K (50000) and shear modulus *G* (50000) when traversed by a moving concentrated force.

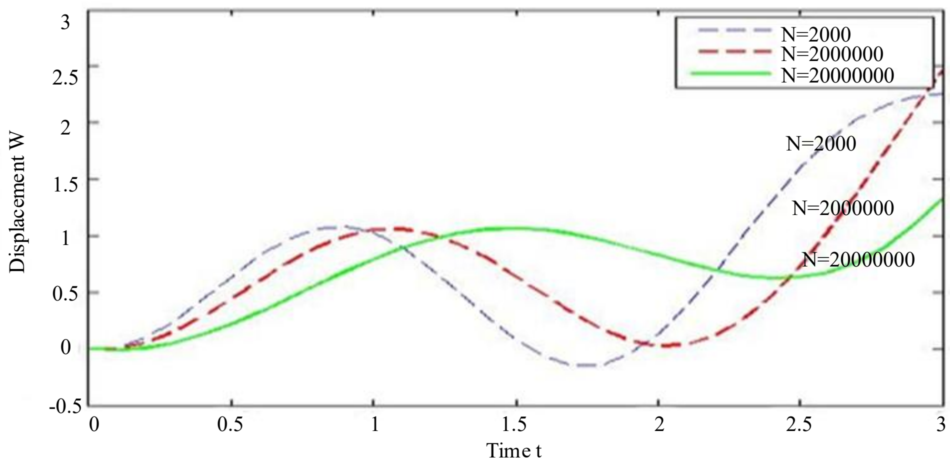

Figure 17. Transverse displacement of a clamped-clamped uniform Bernoulli-Euler beam for varying values of the axial force *N* with fixed parameters: damping coefficient α (2), foundation modulus K (50000) and shear modulus *G* (50000) when traversed by a moving concentrated mass.

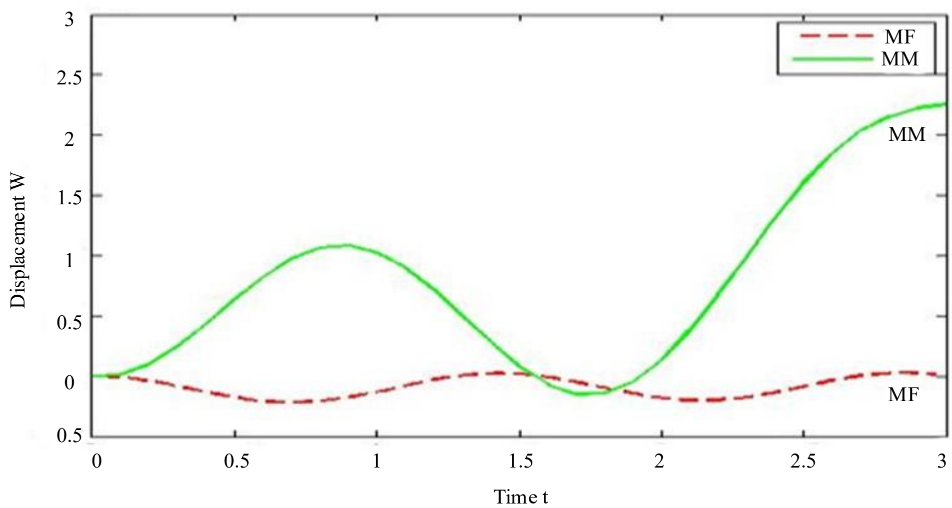

Figure 18. Comparison of the transverse displacement of a clamped-clamped uniform Bernoulli–Euler beam when traversed by a moving concentrated force and a moving concentrated mass.

Similarly, the beam traversed by a moving concentrated mass reaches a state of resonance whenever, $\alpha_{mm} = \theta_m = \frac{\lambda_m c}{L}$

Thus, the moving force and the moving mass problem of the same beam will attain a state of resonance for the same critical speed $\theta_m = \frac{\lambda_m c}{L}$. However, from Eq. (46), α_{mf} and α_{mm} are related by,

$$\alpha_{mm} = \alpha_{mf} - \left(\frac{{\alpha_{mf}}^2 + C^2 U_3(m,t) - \in A_2}{2\,\alpha_{mf}}\right) \qquad (46)$$

From Eq. (46) $\alpha_{mm}$is smaller than α_{mf} which shows that resonance is reached earlier for the moving mass problem than for the moving force problem. We note that the deflection profiles of the comparison graphs above also revealed this same scenario. Therefore, the moving force problem cannot be used as a safe approximation for the moving mass problem in the design of the dynamical system.

CONCLUSION

The study showed that increases in the values of the vital parameters decreased the transverse displacement of the beam for both moving concentrated force and moving mass cases, thereby ensuring its safety, stability, and performance. In addition, the simply supported condition generally had a higher deflection for both the moving force and the moving mass problems. However, this higher deflection was more noticeable for the simply supported condition in the damping case for the moving force with reference to the moving mass. Furthermore, the moving concentrated force problem was not a safe approximation for the moving concentrated mass problem. The results herein were found to agree with those in the literature.

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