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Computation of the Necking in Bars Without Geometrical Imperfection

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Abstract

This paper presents a finite element analysis to compute the neck profiles of smooth, circular cylindrical bars in shear-free grip under uniaxial tensile load. The analysis uses 4-node quadrilateral element with linear displacement variation, as well as 8-node Serendipity quadrilateral element with curved sides and a quadratic variation of the displacement field, for power law hardening materials that follow von Mises plasticity. All the elements are based on isoparametric formulation. The analysis computes the reduction of area for given elongation and the results are compared with the data of percentage elongation and reduction of area (at fracture) for a few materials as available in literature to find close correspondence. This analysis simulates the neck profiles by implementing only that the elements away from the neck stop deforming.

Keywords: Ductility, elongation, reduction-of-area, von Mises plasticity, finite element analysis.

INTRODUCTION

A tension test evaluates two kinds of fundamental mechanical properties of a material, viz., strength properties, such as yield strength and ultimate tensile strength, and ductility proper-ties, such as percent elongation and reduction-of-area. In the tensile test the specimen, round bar or thin or thick sheet of specified size, is subjected to continually increasing uniaxial load by gripping the specimen at its two ends and pulling it, whereby the specimen elongates and ultimately fractures. A brittle material fractures by separation normal to the tensile force, whereas a ductile material necks before fracture.

The percentage elongation and reduction-of-area are obtained by measuring the final length L_f and section area A_f at fracture. Among the ductility parameters, 'elongation' depends on the strain-hardening capacity of the material whereas 'reduction-of-area' is more a measure of the deformation required to produce fracture, and these two cannot be calculated from each other due to the occurrence of necking. The chief contribution to the reduction-of-area results from the necking process. A change in the reduction in area can detect quality change in the material as this is the most structure-sensitive ductility parameter, [1, p. 295]. The materials are called ductile, brittle and notch-brittle as the reduction in area is >50%, <10% and moderate (say, 30%), [2, p. 4.6].

The necking is a unique phenomenon and has given rise to curiosity and critical study, both experimental and analytical, touching upon complexities involved in the analysis. It is first shown in Chen [3] that the calculated necked profiles obviously resemble the shapes observed in tests, but no precise comparison is made. The analysis makes

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remarks that the results are far from conclusive and improvements of the numerical technique will perhaps lead to a better understanding of the cup-cone type fracture.

After referring to earlier investigations, e.g., Chen [3], Needleman [4], Norris *et al.* [5], Saje [6] stresses that all reached more or less different results, and that the present analysis comes to again new results. A chain of further analysis, including the studies of Wriggers and Simo [7], Wriggers *et al*. [8], Miehe *et al*. [9] shows the development of neck in bars without test comparisons. Valiente [10] concluded the finite element solution with the finding of significant differences with the Bridgman's solution with regard to computation of load in the simulated tensile test.

The present finite element analysis aims to compute the necked profile of bars for five different materials and compare the results with the reduction-of-area values available in literature. The prediction of the extent of necking in a shear-free end specimen is evidently dependent upon the initial geometrical imperfection used in order to initiate necking, and therefore, attempt is made in the present analysis to develop necking in a smooth bar without initial imperfection. For the analysis, material properties, viz., Poisson's ratio, ν, Young's modulus, E, yield strength, s_Y, and ultimate tensile strength, s_u, are made use of. As the finite element analysis bases the calculation on the term $[B]^T[D^{ep}][B]$ in the governing equation, the present article attempts to make it explicit in the following two sections how the term develops and is used. The tensile test simulation appears in the penultimate section, before the conclusion.

GOVERNING EQUATION

The condition of equilibrium of a material body with volume V and surface area S is described as

$$\int \sigma_{ij} n_j dS + \int F_i dV = 0 \tag{2.1}$$

where σ_{ij} is the symmetric stress tensor, n_j are the direction cosines, and F_i the body forces per unit volume [11, p. 71]. By applying Gauss's divergence theorem that relates surface integral to volume integral, Eq. (2.1) transforms to

$$\int \left(\frac{\partial \sigma_{ij}}{\partial x_j} + F_i\right) dV = 0 \tag{2.2}$$

Where x_j are the coordinate axes with j =1,2,3. For arbitrary volume, then

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \tag{2.3}$$

which using von Karman's notation becomes

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0 \end{aligned} \tag{2.4}$$

where the stress tensor being symmetric, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$. By employing the material constitutive law, i.e., stress-strain relation

$$\{\sigma\} = [D]\{\epsilon\} \quad ([D] \text{ is the elastic constitutive matrix}) \tag{2.5}$$

which in the plane stress case ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$ and $\gamma_{yz} = \gamma_{zx} = 0$) takes the form

$$\underbrace{\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}}_{\{\sigma\}} = \underbrace{\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}}_{[D]} = \underbrace{\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}}_{\{\epsilon\}}, \tag{2.6}$$

where E and ν are Young's modulus and Poisson's ratio, eq.(2.4) in two dimension turns

$$\frac{E}{1-\nu^2}\left[\frac{\partial}{\partial x}\left(\epsilon_x + \nu\epsilon_y\right) + \frac{\partial}{\partial y}\left(\frac{1-\nu}{2}\gamma_{xy}\right)\right] = -F_x,$$
$$\frac{E}{1-\nu^2}\left[\frac{\partial}{\partial x}\left(\frac{1-\nu}{2}\gamma_{xy}\right) + \frac{\partial}{\partial y}\left(\nu\epsilon_x + \epsilon_y\right)\right] = -F_y \tag{2.7}$$

Further, for geometrically linear or small deflection problems the strain components are related to the displacements u, v in x, y directions, as

$$\epsilon_x = \frac{\partial u}{\partial x}, \ \epsilon_y = \frac{\partial v}{\partial y}, \text{ and } \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{2.8}$$

Upon substituting ϵ_x, ϵ_y, and γ_{xy} in terms of u and v from Eq. (2.8) in Eq. (2.7) and rearranging, we get a pair of simultaneous partial differential equations (where the mixed partial derivatives of the second order are equal, [12, p. 653]) as

$$\frac{E}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 v}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x\partial y}\right) = -F_x$$
$$\frac{E}{1-\nu^2}\left(\nu\frac{\partial^2 u}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2}\right) = -F_y \tag{2.9}$$

which are the governing equations in solid mechanics in two-dimensional plane stress case for the unknowns u and v, [13, p. 1309].

The governing equations are rewritten in the matrix form as

$$\frac{E}{1-\nu^2}\begin{bmatrix} \left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial y^2}\right) & \left(\nu\frac{\partial^2}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x\partial y}\right) \\ \left(\nu\frac{\partial^2}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x\partial y}\right) & \left(\frac{\partial^2}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2}\right) \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} -F_x \\ -F_y \end{Bmatrix} \tag{2.10}$$

FINITE ELEMENT SOLUTION

If the displacement field $\begin{Bmatrix} u \\ v \end{Bmatrix}$ be represented in the form

$$u = [N_j]\{u\} \text{ and } v = [N_j]\{v\} \tag{3.1}$$

where $[N_1\, N_2\, N_3\, N_4]$, i.e., $[N_j]$ are the interpolation or shape functions, for four-node element, and $N_1 = (1 - x/a)(1 - x/b)$, $N_2 = (1 - x/a)(y/b)$, $N_3 = (x/a)(y/b)$, $N_4 = (x/a)(1 - y/b)$ and u, v represent the displacement in the coordinate directions x and y of the nodes of the rectangular element with sides a and b ([14, pp. 26-27]) constituting the deforming body, Eq. (2.10) takes the form

$$\frac{E}{1-\nu^2}\begin{bmatrix} \left(\frac{\partial^2 N_j}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial y^2}\right) & \left(\nu\frac{\partial^2 N_j}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial x\partial y}\right) \\ \left(\nu\frac{\partial^2 N_j}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial x\partial y}\right) & \left(\frac{\partial^2 N_j}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial x^2}\right) \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} -F_x \\ -F_y \end{Bmatrix} \tag{3.2}$$

since from Eq. (3.1), $u = [N_j]\{u\}$ and $v = [N_j]\{v\}$.

The solution of the governing partial differential equations is attempted by multiplying Eq. (3.2) by the shape functions $\{N_i\}$ (i = 1, 2, 3, 4) and then using integration by parts [15, pp. 428, 573-574], [14, pp. 18-19]. The method is the Galerkin method [16, pp. 89, 91], or Bubnov-Galerkin method [17, p. 240], [18, p. 50]. Accordingly, referring to [14, pp. 18-19, 29], [16, pp .91-93], and [18, pp. 765-766], it follows

$$\frac{E}{1-\nu^2}\int_0^a\int_0^b \begin{bmatrix} N_i\left(\frac{\partial^2 N_j}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial y^2}\right) & N_i\left(\nu\frac{\partial^2 N_j}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial x\partial y}\right) \\ N_i\left(\nu\frac{\partial^2 N_j}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial x\partial y}\right) & N_i\left(\frac{\partial^2 N_j}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 N_j}{\partial x^2}\right) \end{bmatrix} dx\, dy \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$= -\begin{Bmatrix} \int_0^a \int_0^b N_i F_x \, dx \, dy \\ \int_0^a \int_0^b N_i F_y \, dx \, dy \end{Bmatrix}$$

$$\text{or, } \frac{E}{1-\nu^2}\int_0^a \int_0^b \begin{bmatrix} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) & \left(\nu\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\right) \\ \left(\nu\frac{\partial N_i}{s\partial y}\frac{\partial N_j}{\partial x} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right) & \left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right) \end{bmatrix} dx\, dy \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$+ \frac{E}{1-\nu^2}\oint \begin{bmatrix} \left(N_i\frac{\partial N_j}{\partial x}n_x + \frac{1-\nu}{2}N_i\frac{\partial N_j}{\partial y}n_y\right) & \left(\nu N_i\frac{\partial N_j}{\partial y}n_x + \frac{1-\nu}{2}N_i\frac{\partial N_j}{\partial x}n_y\right) \\ \left(\nu N_i\frac{\partial N_j}{\partial x}n_y + \frac{1-\nu}{2}N_i\frac{\partial N_j}{\partial y}n_x\right) & \left(N_i\frac{\partial N_j}{\partial x}n_x + \frac{1-\nu}{2}N_i\frac{\partial N_j}{\partial y}n_y\right) \end{bmatrix} d\Gamma \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{3.3}$$

$$= -\begin{Bmatrix} \int_0^a \int_0^b N_i F_x \, dx \, dy \\ \int_0^a \int_0^b N_i F_y \, dx \, dy \end{Bmatrix}$$

$$\text{or, } \frac{E}{1-\nu^2}\int_0^a \int_0^b \begin{bmatrix} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) & \left(\nu\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\right) \\ \left(\nu\frac{\partial N_i}{s\partial y}\frac{\partial N_j}{\partial x} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right) & \left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + \frac{1-\nu}{2}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right) \end{bmatrix} dx\, dy \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

where the contour integral (i.e., derivative boundary condition, [19, pp. 91, 115-125]) over the boundary Γ is assumed zero.

The final shape to the last equation follows as

$$\frac{E}{1-\nu^2}\int_0^a \int_0^b \begin{bmatrix} \left(\frac{\partial N_i}{\partial x}\right) & \left(\frac{\partial N_i}{\partial x}\nu\right) & \left(\frac{\partial N_i}{\partial y}\frac{1-\nu}{2}\right) \\ \left(\frac{\partial N_i}{\partial y}\nu\right) & \left(\frac{\partial N_i}{\partial y}\right) & \left(\frac{\partial N_i}{\partial x}\frac{1-\nu}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix} dx\, dy \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

$$\text{or, } \int_0^a \int_0^b \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial y} \\ 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \left(\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \right) \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix} dx\, dy \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} \tag{3.4}$$

or $\int_0^a \int_0^b \; [B]^T [D][B] \, dx\, dy$ {U} = {F}

where, $B = \begin{bmatrix} \frac{\partial N_j}{\partial x} & 0 \\ 0 & \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} \end{bmatrix} = [L][N]$

is the strain-nodal displacement matrix, $[B]^T$ is the transpose of B-matrix, and [D] is the constitutive matrix as in the Eq. (2.6).

In case of solids of revolution subjected to axisymmetric loading, as applicable for round section tension specimen, the Eq. (3.4) gets the equivalent form in cylindrical coordinate system as, [14, pp. 30, 57],

$$2\pi \int_0^a \int_0^b \; [B]^T [D][B] r \, dr\, dz \; \{U\} = \{F\} \tag{3.5}$$

where the strain-displacement matrix, L, is now defined as

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{Bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{3.6}$$

$$Or\ \{\epsilon\} = [L] \{u\}$$

and the stress-strain matrix, D, is

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \\ \sigma_\theta \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{Bmatrix} (1-\nu) & 0 & 0 & \nu \\ \nu & (1-\nu) & 0 & \nu \\ 0 & 0 & (1-2\nu)/2 & 0 \\ \nu & \nu & 0 & (1-\nu) \end{Bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_{rz} \\ \gamma_\theta \end{Bmatrix} \tag{3.7}$$

$$Or\ \{\sigma\} = [D] \{\epsilon\}$$

The final integral term in Eq. (3.4) or Eq. (3.5) is the 'element stiffness matrix', [k]. Instead of a single rectangular element constituting the region of solution as above, a number of elements, joined at the nodes, each with its stiffness matrix[k], together build the solution area. The elements are, however, now described in local coordinates, (ξ, η), normalized and varying from -1 to +1, instead of in the global coordinates (x, y). The shape functions in the local coordinates are as given in Bathe [20, p. 200]. The same shape functions to relate the global and local coordinates give rise to isoparametric elements, used in the present analysis.

The elements' stiffness are assembled by following the procedure, say, as given in Smith [14, pp. 46-48, 62-65]. In the present analysis, specified displacement boundary conditions (i.e., Dirichlet or essential boundary condition) are incorporated through penalty method, i.e., 'big spring' technique, [20, pp. 111-113, 141], [14, pp. 48, 160]. The integration for each quadrilateral element stiffness is performed numerically using Gaussian quadrature.

Elasto-plastic Problem Solution

In elasto-plastic strain-hardening material, the constitutive incremental stress-strain relation is given by, [11, pp. 267-270], [14, p.159], [21, pp. 225-229],

$$\{d\sigma\} = [D^{ep}]\{d\epsilon\}$$

$$= \left[[D] - \frac{[D]\frac{\partial Q}{\partial\{\sigma\}}\left(\frac{\partial F}{\partial\{\sigma\}}\right)^T [D]}{A + \left(\frac{\partial F}{\partial\{\sigma\}}\right)^T [D]\,\frac{\partial Q}{\partial\{\sigma\}}} \right] \{d\epsilon\} \tag{3.8}$$

where $[D^{ep}]$ and [D] are the elasto-plastic and elastic stress-strain matrix respectively, Q and F are the flow surface and yield ssurface, and A is the shift of the yield surface.

With fully associated flow, $Q \equiv F$ (the vector of plastic strain increment is normal to the yield surface), and Huber-von Mises yield surface given by $(3J_2)^{1/2} - Y = 0$, where $J_2 \left[= 1/6\left(\left\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right\} + {\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{xz}}^2\right)\right]$ is the second invariant of stress deviator tensor, $\bar{\sigma}$ effective stress = $(3J_2)^{\frac{1}{2}}$, and Y uniaxial stress at yield, $\partial F/\partial\{\sigma\}$ is given as, [18, p. 464],

$\frac{\partial F}{\partial\{\sigma\}} = \frac{\partial F}{\partial J_2} \cdot \frac{\partial J_2}{\partial\{\sigma\}} = \frac{\sqrt{3}}{2\sqrt{J_2}} \cdot \frac{\partial J_2}{\partial\{\sigma\}} = (3/2\bar{\sigma}) \cdot \frac{\partial J_2}{\partial\{\sigma\}}$, and therefore,

$$\frac{\partial F}{\partial \sigma_x} = (3/2\bar{\sigma}).\frac{\partial J_2}{\partial \sigma_x} = (3/2\bar{\sigma}).\frac{2\sigma_x - \sigma_y - \sigma_z}{3}, \frac{\partial F}{\partial \sigma_y} = (3/2\bar{\sigma}).\frac{\partial J_2}{\partial \sigma_y} = (3/2\bar{\sigma}).\frac{2\sigma_y - \sigma_z - \sigma_x}{3},$$
$$\frac{\partial F}{\partial \sigma_z} = (3/2\bar{\sigma}).\frac{\partial J_2}{\partial \sigma_x} = (3/2\bar{\sigma}).\frac{2\sigma_z - \sigma_x - \sigma_y}{3}, \frac{\partial F}{\partial \tau_{xz}} = (3/2\bar{\sigma}).\frac{\partial J_2}{\partial \tau_{xz}} = (3/2\bar{\sigma}).(2\tau_{zx}), \tag{3.9}$$

where in terms of axisymmetric stresses, $\sigma_r \equiv \sigma_x$, $\sigma_z \equiv \sigma_z$, $\tau_{rz} \equiv \tau_{xz}$, and $\sigma_\theta \equiv \sigma_y$.

The term A in Eq. (3.8) is the plastic modulus, i.e., local slope of the uniaxial stress-plastic strain curve, [21, p. 229]. For power law hardening model, $\sigma_1 = K\,\epsilon_n^1$, it is expressed as

$$A = \frac{E.\frac{d\sigma_1}{d\epsilon_1}}{E - \frac{d\sigma_1}{d\epsilon_1}} = \frac{E.(K\,n\,\epsilon_1^{n-1})}{E-(K\,n\,\epsilon_1^{n-1})} = \frac{E.(K\,n\,\bar{\epsilon}_1^{n-1})}{E-(K\,n\,\bar{\epsilon}_1^{n-1})} \tag{3.10}$$

extrapolating the material behaviour determined under uniaxial state of stress to $\bar{\sigma} = K\bar{\epsilon}^n$, more general state of stress, viz., 'effective stress', $\bar{\sigma}$, using invariant function 'effective (or equivalent) strain', $\bar{\epsilon}$, defined (for axisymmetric state of stress) as

$$\bar{\epsilon} = \frac{\sqrt{2\{(\epsilon_r - \epsilon_\theta)^2 + (\epsilon_\theta - \epsilon_z)^2 + (\epsilon_z - \epsilon_r)^2 + 3\gamma_{rz}}}{3} \tag{3.11}$$

The value of A is calculated from the known material properties K and n, the hardening modulus and exponent respectively, and Young's modulus. E.

The finite element governing equation for elastic-plastic analysis replaces $[D]$ by $[D^{ep}]$ in the stiffness matrix, $[k]$, in Eq.(3.4) or Eq.(3.5), applicable for four-as well as eight-node element analysis. It is a nonlinear equation, of the nodal displacement, $\{U\}$, because of the non linear relationship, Eq. (3.8), between the stress σ and the strain ϵ. To solve the governing equation, iterative method is employed. Moreover, an incremental analysis is used, i.e., the external load is applied in increments, m-th, (m+1)-th etc. The displacement increment of the i-th iteration, $\{\Delta U\}^{(i)}$ [= ${}^{m+1}\{U\}^{(i)}$ - ${}^{m+1}\{U\}^{(i-1)}$ where ${}^{m+1}\{U\}^{(i-1)}$ is known from the previous iteration, and ${}^{m+1}\{U\}^{(0)} = {}^{m}\{U\}$], is obtained, [14, pp. 148, 138-139], [11, pp. 299-303], [22, pp. 573-574], by solving the equation,

$${}^1[K]\{\Delta U\}^{(i)} = {}^{m+1}\{F\} + \text{internal nodal forces} \tag{3.12}$$

where the (system) stiffness matrix ${}^1[K]$ is the elastic stiffness as used in the first load increment, which is the 'initial stress' iterative procedure.

Calculation of the internal nodal forces of the elements begins by looking at their `centres' (with local coordinates $\xi = \eta = 0$) and involves two steps, viz., first, from the displacement increment obtained from an elastic solution, estimate of the strain increments at the 'centre' is made (via the $[B]$ matrix), as also stress increments are calculated from the strain increments, second, using the strain increments obtained at the first (step) elastic calculation, a revised stress increments are calculated, from Eq. (3.8), which do not balance the stress increments made in the first step.

The unbalanced stresses $\{\sigma_0\}$ give the internal nodal forces by numerical integration of $2\pi\iint [B]^T\{\sigma_0\}rdr\ dz$. The unbalanced internal nodal forces are then distributed, i.e., $\{\Delta U\}^{(i)}$ is calculated, using the Eq. (3.12). The procedure continues till the ratio of $|\{\Delta U\}^{(i)}|$ and $|{}^{m+1}\{U\}^{(i-1)}|$ is less than or equal to the set tolerance.

TENSILE TEST SIMULATION

In the present analysis the round cylindrical bar is axisymmetric with respect to the z-axis and symmetric with respect to the $z = 0$ plane. Because of symmetry, only one quarter of the specimen is discretised. The initial half-length to radius, $\frac{\left(\frac{L_0}{2}\right)}{R_0}$, or length to diameter, $\frac{L_0}{D_0}$, ratio is 4. The quarter bar is discretised into 36 (12 rows $\times$ 3 columns) elements which are isoparametric, either 4- or 8-node quadrilaterals. The nodes on the $z = 0$ as well as $r = 0$ axis are constrained to move along the respective axis and the node at the origin is fixed.

The bar is initially smooth and is deformed by imposing axial displacement at the free end which is unrestrained of lateral movement, that is, shear-free. The axial displacement is applied in four increments, the first, similar to that in Argyris and Chan [23, p. 163] and Tu˘gcu and Neale [24, p. 1081], taken near the point of initial yielding, and the rest of the increments equally divided. At each increment, except the last one, the analysis is carried out for all elements. The elastoplastic analysis is based on J_2-flow theory, Eq. (3.8), and the iterative scheme of Eq. (3.12). A uniaxial behavior of power law hardening,$\sigma_1 = K\epsilon_I^n$, is assumed to apply.

Usually, the data for K and n are scarce. An expression for K can be, see Tu˘gcu *et al.* [25],

$K = \sigma_Y^{I-n} E^n$ or, $K = E^n s_Y^{I-n}$ (since, $\sigma_Y \cong s_Y$, yield strength), (4.1)

and, s_u, ultimate tensile strength, can be expressed as, see Dieter [1, pp. 722], $s_u = K\left(\frac{n}{e}\right)^n$ where e = 2.718, which lead to the equation

$$\frac{s_u}{s_Y} = \left(\frac{E}{e\, s_Y}\right)^n n^n \quad (4.2)$$

So that n and K are calculated in the present analysis using Eq. (4.2) and Eq. (4.1) and readily available values of the yield strength and ultimate tensile strength of the test material. The value of n is, however, assumed zero (and K equal to s_Y) when $(s_u - s_Y)/s_Y$ is less than 10%.

During a tensile test, up to the maximum load point the cross-sectional reduction over the entire gauge length is uniform. Thereafter, an instability sets in, and necking commences since the increase in strength due to strain hardening is unable to compensate the decrease in load bearing capacity as a result of reduced cross-sectional area. As the neck develops, the plastic deformation is localised in the neck region and the material elements away from the neck stop deforming, [26, 27].

Therefore, in the last increment of the present analysis, the Eq. (3.12) is solved only forthe neck elements. The neck elements are assumed to be the single row of elements with z = 0 nodes. This, in effect, keeps the elements away from the neck out of the deformation process since $\{\Delta U\}^{(i)}$ is calculated for only the neck elements which deform. This new procedure gives rise to the usual hour-glass shape of the neck as develops in the actual necking process. It remains to be seen how closely the simulated necks compare with those observed in tests.

The present analysis has adapted the finite element programs of Smith [14] with changes to include the isotropic hardening of the materials as well as incorporated the mesh generation program of Segerlind [28] with suitable modifications.

Simulation Compared with Test Deformation

For the purpose of comparison, test data as given in literature for five materials, viz., (1) cold drawn stress relieved 1144 steel bars subjected to medium draft, (2) martensitic stainless steel, annealed and cold drawn, (3) Ti-6Al-4V at 24°C, (4) steel 5046, longitudinal, and (5) steel 1137, tempered 650°C with their mechanical properties are listed in the Table 1.

The data are for 25 mm diameter and 100 mm length bar except for the material 5 which is for 12.8 mm diameter and four times length. The computation adopts a 4- and 3-point Gaussian integration for the 8- and 4-node elements respectively. The number of iteations does not exceed seventeen for the set

Table 1. Comparison of the simulated and test percent reduction-of-area.

Material 1†		**Material 2**‡	**Material 3**§	**Material 4**¶	**Material 5***
Poisson's ratio, ν	0.3^b	0.3^b	0.3	0.3^b	0.3^b
Young's modulus, E, GPa	207.0^b	207.0^b	110.0	207.0^b	207.0^b
Yield strength, s_Y, MPa	689.0	690.0	825.0	585.0	483.0
Tensile strength, s_u, MPa	827.0	760.0	890.0	820.0	655.0
Hardening exponent, n [calculated from eq. (4.2)]	0.083	0.055	0.0	0.123	0.108
Hardening modulus, K, MPa [calculated from eq. (4.1)]	1106.0	941.2	825.0	1200.5	927.1
Elongation	0.10	0.14	0.14	0.255	0.28
Reduction-of-area, %	30	40	41	64.1	69
-do- [calculated for 8-node element bar]	28.7	39.0	39.2	63.7	68.4
-do- [calculated for 4-node element bar]	28.8	39.1	39.3	64.0	68.6

†*[29, p. 244]* ‡*[30, pp. 156-158]* §*[31, p. 630]* ¶*[29, p. 357]* **[32, pp. 308-310]* ^{b}*[29, p. 393]*

tolerance of 0.001. Chen and Mizuno [22, pp. 575-576] finds the tolerance of the order of 10^{-4} or 10^{-5} suggested by Nayak and Zienkiewicz [33] rather restrictive, and reports obtaining good results using a displacement convergence tolerance of the order of 10^{-2} to 10^{-3} which is adopted in the present analysis.

As to the number of integration points, Argyris *et al*. [34] report using one integration point for four-node quadrilateral elements and four integration points for eight-node quadrilaterals, and Aravas [35] uses 2×2 Gauss integration for eight-node axisymmetric elements, and Lee and Zhang [36] employ 2×2 integration points for both four- and eight-node isoparametric elements, whereas Lu and Levy [37] make use of eight-node isoparametric elements with four Gauss points. As for triangular elements, Tu˘gcu [38] adopts a four point Gaussian integration stating that this choice is arbitrary as no optimization is undertaken for the presented application, and Argyris *et al*. [34] use one integration point, whereas Neale and Tu˘gcu [39] usea seven-point Gaussian integration scheme but the same investigators employ one-point scheme in [24] and [40] and four-point integration scheme in [41].

From the small sample above it is unclear what criteria decide the number of Gauss integration points. In the present study Gauss points are so chosen that the results from the four-node and eight-node analysis are with smallest difference and vary little from the literature values.

It is seen in Table 1 that the test and computed percent area-of-reduction for the eight-and four-node element bars are more or less equal for all materials. The difference between the four-node and eight-node results are less than one percent, whereas the difference between the percent reduction-of-area of the present simulation and the literature test values is less than five percent.

CONCLUSION

The aim of the present work was to simulate the deformation profile of a smooth round cylindrical bar under uniaxial tension with shear-free boundary condition. For this, a von Mises yield surface, associated flow, elastic-power law hardening material model was assumed. It is shown that the implementation of stoppage of deformation at the elements away from the neck, at the last load increment, will yield a hour-glass necking profile. The profiles are found to be close to the reported literature data with respect to the reduction-of-area values at fracture, and the two types of elements, eight-node and four-node, lead to results which differ by less than one percent. To return to Chen [3], the numerical technique initiated in the present analysis to develop necking in a tensile bar will perhaps lead to a better understanding of the fracture phenomena in a tension bar.

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