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Research Paper

Optimal Homotopy Analysis Method (OHAM) For the Approximate Series Solution of Non-linear Partial Differential Equation

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Abstract**—**In this article, we have used the Optimal Homotopy Analysis Method (OHAM), which a basically semi analytic method to solve the differential equations. The ability for the user to choose the convergence control parameter, auxiliary linear operator, auxiliary function, and starting approximation is what sets apart the OHAM technique. We guaranteed the efficacy and efficiency of the procedure by fine-tuning the convergence control parameter. We solved a non-linear partial differential equation (PDE) using OHAM to show off its capabilities, and we verified that the series solution converged. Optimal homotopy analysis technique each person has complete discretion over selecting the auxiliary function, auxiliary linear operator, convergence control value, and beginning approximation. We use a non-linear partial differential equation (PDE) to demonstrate the power of OHAM. We guarantee the reliability and quick convergence of the series solution obtained from OHAM by carefully adjusting the convergence control parameter. Our findings confirm that the approach can yield precise and convergent solutions. Because the user may adjust the auxiliary function, linear operator, convergence parameter, and initial guess, OHAM offers a unique benefit that allows the method to be specifically tailored to the features of the differential equation being solved. In this method we have optimized the convergence control parameter c_0.Also we have compare the obtain result with exact solution to check the efficiency of the method. With the assurance of the series solution's convergence, we had used this method to solve non-linear partial differential equations (PDEs).

Keywords—OHAM, PDE, Series solution, Maclaurin series, Convergence Control Paramete, square residual error.

I. INTRODUCTION

The solution of physical and other problems involving functions of several variables, such as the propagation of sound or heat, fluid flow through porous media, ground water flow, electrostatics, electrodynamics, and many more, is achieved mathematically through the use of partial differential equations (PDE) [1], which contain partial derivative terms. A partial differential equation comprises of an unknown function, an unknown function that depends on numerous independent variables, and partial derivatives of the unknown function with regard to the independent variables.

Partial differential equations [2-4] are universal throughout Mathematics, Sciences and engineering. A number of the most significant physical theories, including hydrodynamics, elasticity, quantum mechanics, relativity, and non-relativistic theory, are outlined quantitatively in these texts. Beyond the purview of partial differential equations, the supplemental current relativistic quantum field theories in theory result in equations with an infinite number of unknowns.

In theoretical physics, classical field theory is foundational, fundamentally expressed through systems of partial differential equations (PDEs). These PDEs underpin quantum field theory, which builds on classical concepts to describe the behavior of particles and fields at the quantum level.

The Standard Model of particle physics, grounded in Yang-Mills-Higgs field theory, offers a thorough framework for understanding the fundamental forces and particles. It specifically addresses the weak and strong interactions, two of the four fundamental forces in nature. This model relies on gauge theories, which are intricately linked to the mathematics of PDEs, to elucidate particle interactions and the resulting observable phenomena.

Moreover, differential equations are essential throughout physics. Nearly every physical theory and model employs them to describe system dynamics. For example, Classical Mechanics is described by ordinary differential equations (ODEs), which can be viewed as one-dimensional special cases of PDEs. These ODEs explain the motion of objects under various forces and constraints, playing a crucial role in the study of macroscopic systems.

Thus, whether dealing with the complex PDEs of field theories or the more straightforward ODEs of classical mechanics, differential equations provide the mathematical framework essential for describing and predicting physical phenomena across a broad range of scales and contexts.
The articles of the compendium that contain the Maxwell, Yang-Mills, Einstein, Euler, and Navier-Stokes equations are cited as instances of partial differential equations that support some of our most fundamental physics theories.

II. Basic idea of Optimal homotopy analysis method (OHAM)

In science, engineering, and finance, perturbation and asymptotic methods are frequently used to produce analytical approximations of linear and non-linear problems. Asymptotic and perturbation techniques are widely known to be very reliant on tiny or big parameters, they are frequently limited to weekly non-linear problems.
Thus, developing some analytical approximation techniques that hold true for strongly non-linear conditions and are entirely independent of small and large parameters is imperative. The Homotopy analysis method (HAM) [4,5], which was put out by Chinese researcher Shijun Liao, is one such type of analytical approximation technique. The homotopy, a key idea in differential geometry and topology, serves as its foundation. The homotopy analysis method is unaffected by physical characteristics, no matter how big or tiny. By selecting an auxiliary parameter c0, also referred to as the convergence control parameter, this method's main benefit is that it gives us a practical technique to ensure the convergence of homotopy series solutions of non-linear problems. The Optimal Homotopy Analysis Method (OHAM) [6-9] is based on Homotopy Analysis Method (HAM) in which the convergence-control parameter c_0 is optimized.
The Optimal homotopy analysis method has boundless freedom to choose the initial approximation, auxiliary function, auxiliary linear operator and a convergence control parameter. However, with such great freedom comes the predicament of deciding just how to proceed. There have been a number of linear and non–linear differential equations [10] to which the Homotopy analysis method has been applied; however, the assortment of initial approximation, auxiliary linear operator,Between equations and authors, there are notable differences in the auxiliary function and convergence control parameter. Researchers can use the homotopy analysis approach to solve a non-linear differential equation with less computational efficiency by selecting different choices for the beginning approximation, auxiliary linear operator, auxiliary function, and convergence control parameter.
Let us consider a general kind of non-linear partial differential equation,

$$\mathrm{N}\left[u(x;t)\right]=0 \qquad (1)$$

x and t indicate the independent variables and N is a non-linear operator when u is an unknown function. For the reason of simplicity, we ignore any beginning or boundary conditions. Initially, we employ the traditional optimal homotopy analysis method to generate the zero order [4] deformation equation as follows:

$$(1-q)\,\mathrm{L}\left[\phi(x,t;q)-u_0(x,t)\right]=c_0 q\,\mathrm{N}\left[\phi(x,t;q)\right] \qquad (2)$$

where c_0 is the auxiliary parameter referred to as convergence control, and $q \in [0,1]$ is the embedding parameter. parameter, is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function and $u_0(x,t)$ is an initial guess of . HAM stands for homotopy analysis method. Our options for linear operators are quite flexible and initial guess. Obviously, when the embedding parameter $q=0$ and $q=1$,it holds $\phi(x,t;0)=u_0(x,t)$ and $\phi(x,t;1)=u(x,t)$ respectively. Thus, as q increases from 0 to 1, the solution

$\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Maclaurin series with respect to q, we have

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m;u_m(x,t)=\frac{1}{m!}\left.\frac{\partial^m\phi(x,t;q)}{\partial q^m}\right|_{q=0} \quad (3)$$

If the auxiliary linear operator, the initial guess and the convergence control parameter c_0 are properly chosen, then the series (3) converges at $q=1$, then we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t), \quad (4)$$

this must be one of solutions of the original non-linear equation, as proved by Liao [4].

Define the vectors

$$u_m=\{u_0(x,t),u_1(x,t),...,u_m(x,t)\} \quad (5)$$

Differentiating the equation (2) m-times with respect to q, dividing by m! and finally setting q = 0, we get the following m order deformation equation

$$\mathrm{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=c_0\delta_m\left[u_{m-1}(x,t)\right] \quad (6)$$

where,

$$\delta_m\left[u_{m-1}(x,t)\right]=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathrm{N}\left[\phi(x,t;q)\right]}{\partial q^{m-1}}\right|_{q=0} \text{ and}$$

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$$

Applying inverse operator both side to equation (6)

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+c_0\mathrm{L}^{-1}\left[\delta_m\left[u_{m-1}(x,t)\right]\right] \quad (7)$$

Solving equation (7), for we obtain the approximate solution of equation (1) as,

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t) \quad (8)$$

Yabushita et al. [11] have used the Homotopy analysis method (HAM) to solve two coupled non-linear Ordinary differential equations. They have used the "optimization method" to find out the optimal convergence control parameters by means of the minimum of the squared residual of governing equation as

$$E_m(c_0)=\iint_{\Omega}\left[\mathrm{N}\left\{\sum_{n=0}^{m}u_n(x,t)\right\}\right]^2 d\Omega$$

(9) (9)

But when the problems are real world, it is difficult to solve double integration of sum of square residual, so we use its approximate sum form as

$$E_m(c_0)=\frac{1}{(N+1)(M+1)}\sum_{i=0}^{N}\sum_{j=0}^{M}\left\{\mathrm{N}\left[\sum_{n=0}^{m}u_n\left(\frac{i}{N},\frac{j}{M}\right)\right]^2\right\}$$

(10)

Equation (10) gives the square residual error at m order approximation. It is obvious that we can find the optimal value of convergence-control parameter c_0 at the any order of approximation. If there exists convergence-control parameter, c_0 for which we get the minimum of the squared residual E_m, is known as the optimal convergence-control parameter c_0.

III. IMPLEMENTATION OF OHAM

Consider the following non-linear homogeneous partial differential equation [12]

$$\frac{\partial u}{\partial t}=u\frac{\partial u}{\partial x};u(x,0)=-x, \quad (11)$$

whose exact solution is $u(x,t)=\frac{x}{t-1};|t|<1$

According to OHAM [5, 6] first we construct the zeroth order deformation equation as,

$$(1-q)\pounds\left[\phi(\mathrm{X},\mathrm{T};q)-S_0(X,T)\right]=c_0 q\mathrm{N}\left[\phi(\mathrm{X},\mathrm{T};q)\right]$$

(12)

where, $q\in[0,1]$ is the embedding parameter, $c_0\ne 0$ is an auxiliary parameter also know as convergence control parameter. We choose $\pounds=\frac{\partial}{\partial t}$ is an auxiliary linear operator and is an initial guess of $u(x,t)$ [13]. According to OHAM expanding $\phi(x,\ t;q)$ in Maclaurin series solved at q, then the corresponding *m*th order deformation equation is can be obtained as

$$\pounds\left[S_m(X,T)-\chi_m S_{m-1}(X,T)\right]=c_0\delta_m\left[S_{m-1}(X,T)\right], \quad (13)$$

where

$$\delta_m\left[u_{m-1}\right]=\left(u_{m-1}\right)_t+\sum_{r=0}^{m-1}u_r\left(u_{m-1-r}\right)_x$$

Taking inverse operator

$$u_{im}(x,t)=\chi_m u_{i(m-1)}(x,t)+c_0\pounds^{-1}\delta_m\left[u_{i(m-1)}(x,t)\right] \quad (14)$$

where

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Solving equation (14), for we obtain the approximate solution of equation (11) as,

$$u(x,t) = \sum_{m=0}^{5} u_m(x,t)$$
$$= \begin{pmatrix} -1 - t - 0.998t^2 - 1.0034t^3 - 09438t^4 \\ -1.4025t^5 \end{pmatrix} x \qquad (15)$$

The approximate solution to the non-linear homogeneous partial differential equation (11) is found in equation (15). Yabushita's [11,14] method allows us to determine the convergence control parameter's ideal value. $c_0 = -1.07$

minimizing the square residual error The error between the exact answer and the approximate solution obtained by OHAM is displayed in table (1) below.

Table 1: The absolute errors of solution of non-linear PDE upto 5thorder approximation by OHAM

(x; t)	|u(OHAM)-u(Exact)|
(0.1,0.1)	0
(0.1,0.2)	1.28E-08
(0.1,0.3)	7.87E-09
(0.1,0.4)	4.18E-08
(0.1,0.5)	1.83E-06
(0.2,0.1)	7.34E-08
(0.2,0.2)	0
(0.2,0.3)	2.50E-07
(0.2,0.4)	3.41E-06
(0.2,0.5)	1.65E-06
(0.3,0.1)	5.64E-07
(0.3,0.2)	1.07E-08
(0.3,0.3)	0
(0.3,0.4)	3.99E-07
(0.3,0.5)	1.50E-08
(0.4,0.1)	1.12E-08
(0.4,0.2)	7.51E-08
(0.4,0.3)	4.00E-08
(0.4,0.4)	0
(0.4,0.5)	1.35E-07
(0.5,0.1)	5.44E-08
(0.5,0.2)	7.71E-06
(0.5,0.3)	3.92E-07
(0.5,0.4)	3.27E-07
(0.5,0.5)	4.22E-07

IV. CONCLUSION

In this study, we concentrated on solving non-linear partial differential equations (PDEs) using the Optimal Homotopy Analysis Method (OHAM). Our research thoroughly investigated the concept of homotopy and the structured construction of deformation equations, from the zeroth order to the m-th order.

A central feature of OHAM is the convergence control parameter, denoted as c_0. This parameter is crucial because it allows for easy monitoring and adjustment of the homotopy-series convergence, distinguishing OHAM from other analytical methods. By carefully adjusting c_0, OHAM ensures the convergence of series solutions to non-linear equations, which is a significant advantage.

We obtained approximate solutions up to the fifth-order approximation and compared these to the exact solutions. This comparison involved calculating the absolute error between the approximate and exact solutions. Our findings, presented in an error table, showed that OHAM provides highly satisfactory solutions to non-linear partial differential equations.

The results from this study highlight the robustness and reliability of OHAM. The method's ability to ensure convergence through the adjustment of the convergence control parameter, combined with its systematic approach to constructing deformation equations, makes it a powerful tool for solving complex non-linear PDEs. The minimal errors observed in our comparisons confirm that OHAM is not only effective but also precise in generating solutions to non-linear problems.

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