

Exact Solutions of Kuramoto–Sivashinsky Equation Using Modified Sub-equation Method

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Abstract

Equations for non-linear evolution are mathematical representations of physical phenomena. The physical interpretation of the solution functions of these equations enhances the viewpoint of many natural processes. Numerous scientists who acknowledge this truth have focused on this area. These mathematical models are even more important since they provide light on a multitude of events when their solutions take on a physical significance. Through their research, numerous scientists create methods and solutions for non-linear evolution equations. In this work, wave solutions of mathematical equations utilized in physics, engineering, and many other applied sciences are generated using a different approach. The Kuramoto–Sivashinsky equation, which has a physically significant role in mathematics, will be covered in this study. The Kuramoto–Sivashinsky equation was successfully created with the modified sub-equation method, which is one of the exact solution production tools in mathematics. Trigonometric and hyperbolic solutions were obtained with this method. These solutions play a significant role for scientists who study shock wave structure and asymptotic behavior. To get at these solutions, numerous laborious and intricate procedures had to be completed. The modern computer technology makes it easy to solve these challenges. The state of the wave at any given time is displayed using three-dimensional, two-dimensional, and contour graphs by assigning unique values to the constants in the found solutions. The technique applied is a practical and trustworthy way to solve non-linear evolution equations. It is a technique that is suggested for figuring out how to solve non-linear evolution equations (NLEEs).

Keywords: Non-linear evolution equations, exact solution, traveling wave solution, the modified sub-equation method, Kuramoto–Sivashinsky equation

INTRODUCTION

Recently, non-linear evolution equations (NLEEs) have been studied in many fields such as chemistry, physics, engineering, plasma physics, and fluid dynamics. Studies are being conducted on issues such as producing exact solutions to the equations studied. Exact solutions were used to solve NLEEs. Many analytical methods, such as the modified Kudryashov method [1], first integral method [2, 3], $(1/G')$ -expansion method [4, 5], (G'/G) -expansion method [6, 7], Sardar sub-equation method [8, 9], and Hirota bilinear method [10, 11], have been used for this purpose.

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The Kuramoto–Sivashinsky (K-S) equation is a powerful and important tool for the mathematical modeling and analysis of complex and chaotic systems. Many studies have examined the K-S equation, and studies in this field are quite extensive. There are many articles and studies on various special cases of this equation, solution methods, and application areas. Some of these studies are in the form of extensive numerical simulation [12], “viscous shocks” and periodic solutions [13], a new topological method [14],

homotopy analysis method [15], detailed crossover analysis [16], Galerkin method [17], Weiss-Tabor-Carnevale method [18] and so on.

Our goal in this work is to use the modified sub-equation method to obtain the exact solution of the K-S problem. One way to express the K-S equation is as follows:

$$u_t - \alpha u^2 u_x - \gamma u_{xx} - \beta u^3 u_x + u_{xxxx} = 0, \quad (1)$$

Here, are the α, γ, β arbitrary constants [19]. Numerous physical phenomena, including phase turbulence, plasma instabilities, and heat dispersion in the reaction-diffusion system, are sources of the K-S equation [13].

MODIFIED SUB-EQUATION METHOD

Let us consider this method to obtain solutions for NLEEs [20]. Consider the NLEEs as

$$\varphi(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0. \quad (2)$$

Applying the wave transmutation

$$u = u(x, t) = U(\xi), \quad \xi = x + vt, \quad v \neq 0, \quad (3)$$

inhere v is the speed of the wave. Equation (2) converts into ODE

$$Q(U, U', U'', \dots) = 0. \quad (4)$$

It is assumed that Equation (4) has a solution

$$U(\xi) = a_0 + \sum_{i=1}^m (a_i \phi^i(\xi) + a_{-i} \phi^{-i}(\xi)), \quad (5)$$

here at least one of the coefficients “ a_m ” is different from zero. The constants to be determined here are a_i , ($0 \leq i \leq m$), and according to the balance principle, $m \in \{1, 2, 3, \dots\}$ is obtained by balancing the term in Equation (4), and the solution of the Riccati equation is $\phi(\xi)$

$$\phi'(\xi) = \mu + (\phi(\xi))^2, \quad (6)$$

Where, μ denotes an arbitrary constant. Specific solutions of the Riccati equation in Equation (6) are given below:

$$\phi(\xi) = \begin{cases} -\sqrt{-\mu} \tanh(\sqrt{-\mu}\xi), & \mu < 0 \\ -\sqrt{-\mu} \coth(\sqrt{-\mu}\xi), & \mu < 0 \\ -\sqrt{-\mu} \tan(\sqrt{-\mu}\xi), & \mu < 0 \\ -\sqrt{-\mu} \tan(\sqrt{-\mu}\xi), & \mu < 0 \\ -\frac{1}{\xi+R}, & \mu = 0 \text{ (Risaccons.)} \end{cases} \quad (7)$$

In Equation (4), applying Equations (5) and (6), setting all coefficients in a_i , ($i = 0, 1, \dots, m$) equal to zero, the new polynomial with respect to $\phi(\xi)$ is obtained according to a non-linear system of algebraic equations resulting in $\phi^i(\xi)$, ($i = 0, 1, \dots, m$). We determine R, v, μ, a_i , ($i = 0, 1, \dots, m$) constants to arrive at solutions of non-linear algebraic equations.

The solutions of Equation (6) are substituted into the constants produced from this system and placed into Equation (5) using the formula in Equation (7). Thus, the exact solutions for Equation (2) are obtained.

SOLUTIONS OF K-S EQUATION

By applying the wave transformation in (3) to the Equation (1),

$$vU' - \alpha U^2 U' - \gamma U'' - \beta U^3 U' + U^{(4)} = 0, \quad (8)$$

is obtained. Where, v represents the wave velocity. Considering the Equation (8), balancing $U^{(4)}$ with $U^3 U'$ gives $m = 1$, and we obtain the following form of the solution:

$$U(\xi) = a_0 + a_1 \phi(\xi) + a_2 \frac{1}{\phi(\xi)}, \quad (9)$$

Where, a_0, a_1, a_2 are the constants to be determined. The following system of algebraic equations can be created using Equation (9) for Equation (8), and the coefficients of the algebraic equation are zero.

$$\begin{aligned} (\phi(\xi))^0: & \quad v\mu a_1 - \alpha\mu a_0^2 a_1 - \beta\mu a_0^3 a_1 - va_2 + \alpha a_0^2 a_2 + \beta a_0^3 a_2 \\ & \quad - \alpha\mu a_1^2 a_2 - 3\beta\mu a_0 a_1^2 a_2 + \alpha a_1 a_2^2 + 3\beta a_0 a_1 a_2^2 = 0, \\ (\phi(\xi))^1: & \quad -2\gamma\mu a_1 + 16\mu^2 a_1 - 2\alpha\mu a_0 a_1^2 - 3\beta\mu a_0^2 a_1^2 - 2\beta\mu a_1^3 a_2 = 0, \\ (\phi(\xi))^2: & \quad va_1 - \alpha a_0^2 a_1 - \beta a_0^3 a_1 - \alpha\mu a_1^3 - 3\beta\mu a_0 a_1^3 - \alpha a_1^2 a_2 - 3\beta a_0 a_1^2 a_2 = 0, \\ (\phi(\xi))^3: & \quad -2\gamma a_1 + 40\mu a_1 - 2\alpha a_0 a_1^2 - 3\beta a_0^2 a_1^2 - \beta\mu a_1^4 - 2\beta a_1^3 a_2 = 0, \\ (\phi(\xi))^4: & \quad -\alpha a_1^3 - 3\beta a_0 a_1^3 = 0, \\ (\phi(\xi))^5: & \quad 24a_1 - \beta a_1^4 = 0, \\ \frac{1}{(\phi(\xi))^1}: & \quad -2\gamma\mu a_2 + 16\mu^2 a_2 + 2\alpha a_0 a_2^2 + 3\beta a_0^2 a_2^2 + 2\beta a_1 a_2^3 = 0, \\ \frac{1}{(\phi(\xi))^2}: & \quad -v\mu a_2 + \alpha\mu a_0^2 a_2 + \beta\mu a_0^3 a_2 + \alpha\mu a_1 a_2^2 + 3\beta\mu a_0 a_1 a_2^2 + \alpha a_2^3 + 3\beta a_0 a_2^3 = 0, \\ \frac{1}{(\phi(\xi))^3}: & \quad -2\gamma\mu^2 a_2 + 40\mu^3 a_2 + 2\alpha\mu a_0 a_2^2 + 3\beta\mu a_0^2 a_2^2 + 2\beta\mu a_1 a_2^3 + \beta a_2^4 = 0, \\ \frac{1}{(\phi(\xi))^4}: & \quad \alpha\mu a_2^3 + 3\beta\mu a_0 a_2^3 = 0, \\ \frac{1}{(\phi(\xi))^5}: & \quad 24\mu^4 a_2 + \beta\mu a_2^4 = 0. \end{aligned} \quad (10)$$

Where, $a_0, a_1, a_2, \mu, \alpha, \gamma, \beta$ and v are constants, aim with a computer package program, find the solutions of the system in Equation (10), and obtain the following stations.

Case 1. If

$$\gamma = \frac{4(9\mu^2 a_0^2 + 2\mu a_2^2)}{a_2^2}, \quad a_1 = 0, \quad \beta = -\frac{24\mu^3}{a_2^3}, \quad \alpha = \frac{72\mu^3 a_0}{a_2^3}, \quad v = \frac{48\mu^3 a_0^3}{a_2^3}, \quad (11)$$

We obtain the following wave solutions by changing the values of Equation (11) into Equation (9) for Equation (1),

$$u_1(x, t) = a_0 - \frac{\coth\left[\sqrt{-\mu}\left(x + \frac{48t\mu^3 a_0^3}{a_2^3}\right)\right] a_2}{\sqrt{-\mu}}. \quad (12)$$

In the hyperbolic solution presented in Equation (12), special values are assigned to the constants, and graphs showing solitary waves at any moment are presented in Figure 1.

Case 2. If

$$\gamma = \frac{4(9\mu^2 a_0^2 + 2\mu a_2^2)}{a_2^2}, \quad a_1 = 0, \quad \beta = -\frac{24\mu^3}{a_2^3}, \quad \alpha = \frac{72\mu^3 a_0}{a_2^3}, \quad v = \frac{48\mu^3 a_0^3}{a_2^3}, \quad (13)$$

We obtain the following wave solutions by changing the values of Equation (13) into Equation (9) for Equation (1),

$$u_2(x, t) = a_0 - \frac{a_2 \tanh \left[\sqrt{-\mu} \left(x + \frac{48t\mu^3 a_0^3}{a_2^3} \right) \right]}{\sqrt{-\mu}}. \quad (14)$$

In the hyperbolic solution presented in Equation (14), special values are assigned to the constants, and graphs showing solitary waves at any moment are presented in Figure 2.

Case 3. If

$$\gamma = \frac{4(9\mu^2 a_0^2 + 2\mu a_2^2)}{a_2^2}, \quad a_1 = 0, \quad \beta = -\frac{24\mu^3}{a_2^3}, \quad \alpha = \frac{72\mu^3 a_0}{a_2^3}, \quad v = \frac{48\mu^3 a_0^3}{a_2^3}, \quad (15)$$

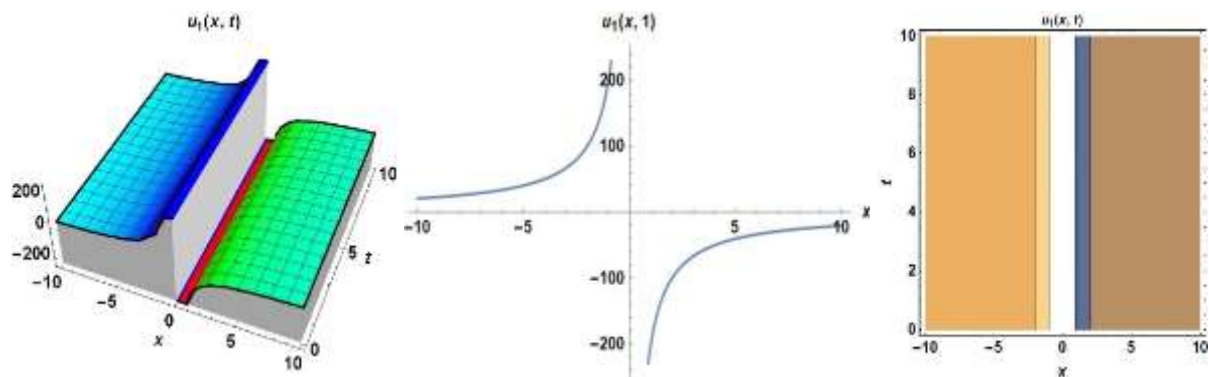


Figure 1. Three-dimensional, two-dimensional, and contour graphs of Equation (12) for $a_0 = 0.1$, $a_2 = 0.2$, $\mu = -0.001$.

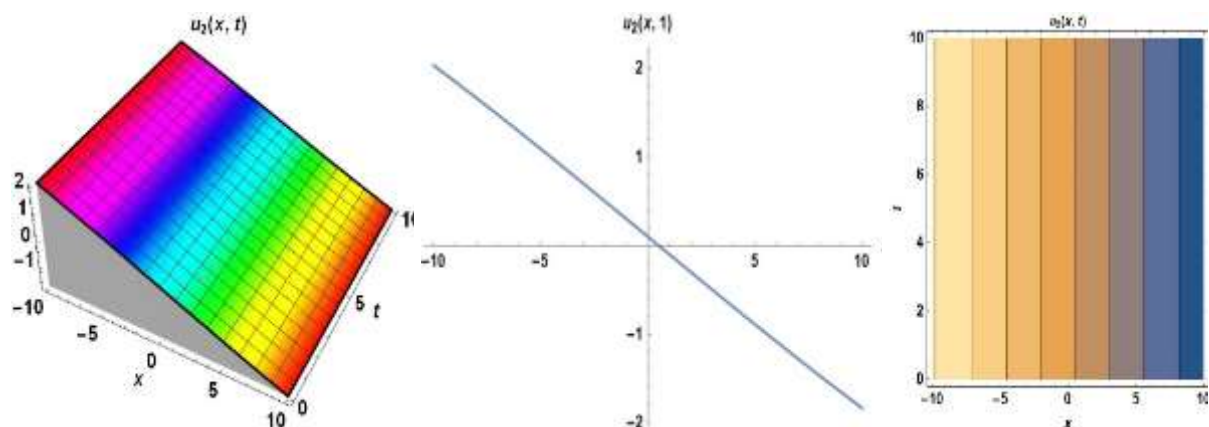


Figure 2. Three-dimensional, two-dimensional, and contour graphs of Equation (14) for $a_0 = 0.1$, $a_2 = 0.2$, $\mu = -0.001$.

We obtain the following wave solutions by changing the values of Equation (15) into Equation (9) for Equation (1),

$$u_3(x, t) = a_0 + \frac{\cot\left[\sqrt{\mu}\left(x + \frac{48t\mu^3 a_0^3}{a_2^3}\right)\right] a_2}{\sqrt{\mu}}. \quad (16)$$

In the trigonometric solution presented in Equation (16), special values are assigned to the constants, and graphs showing solitary waves at any moment are presented in Figure 3.

Case 4. If

$$\gamma = \frac{4(9\mu^2 a_0^2 + 2\mu a_2^2)}{a_2^2}, \quad a_1 = 0, \quad \beta = -\frac{24\mu^3}{a_2^3}, \quad \alpha = \frac{72\mu^3 a_0}{a_2^3}, \quad \nu = \frac{48\mu^3 a_0^3}{a_2^3}, \quad (17)$$

We obtain the following wave solutions by changing the values of Equation (17) into Equation (9) for Equation (1),

$$u_4(x, t) = a_0 - \frac{a_2 \tan\left[\sqrt{\mu}\left(x + \frac{48t\mu^3 a_0^3}{a_2^3}\right)\right]}{\sqrt{\mu}}. \quad (18)$$

In the trigonometric solution presented in Equation (18), special values were assigned to the constants, and graphs showing the solitary wave at any moment are presented in Figure 4.

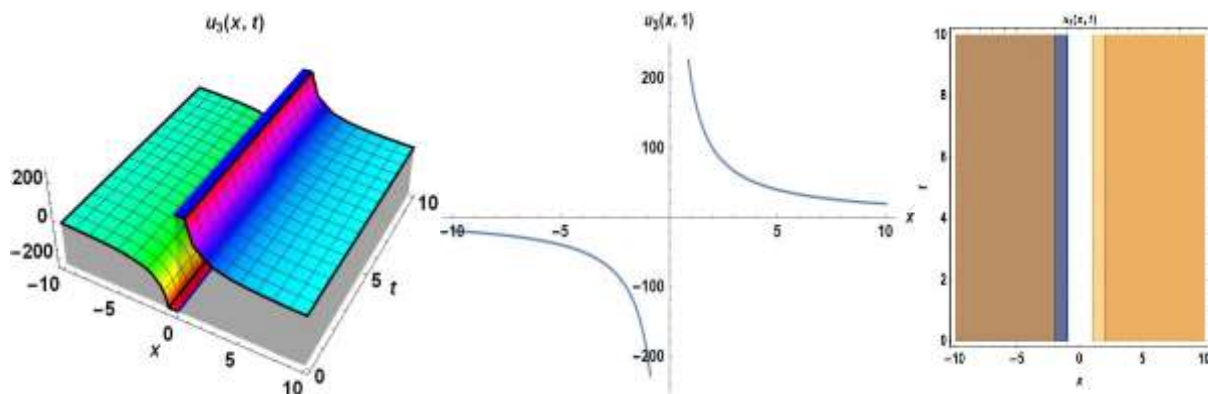


Figure 3. Three-dimensional, two-dimensional, and contour graphs of Equation (16) for $a_0 = 0.1$, $a_2 = 0.2$, $\mu = 0.001$.

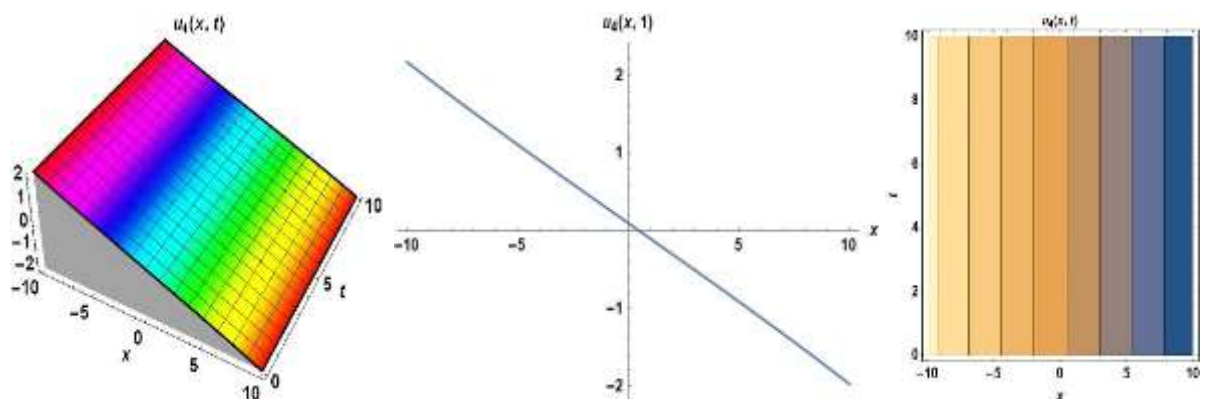


Figure 4. Three-dimensional, two-dimensional, and contour graphs of Equation (18) for $a_0 = 0.1$, $a_2 = 0.2$, $\mu = 0.001$.

Case 5. If

$$\gamma = \frac{4(9a_0^2 + 2\mu a_1^2)}{a_1^2}, \quad a_2 = 0, \quad \beta = \frac{24}{a_1^3}, \quad \alpha = -\frac{72a_0}{a_1^3}, \quad v = -\frac{48a_0^3}{a_1^3}, \quad (19)$$

We obtain the following wave solutions by changing the values of Equation (19) into Equation (9) for Equation (1),

$$u_5(x, t) = a_0 - \sqrt{-\mu} a_1 \tanh \left[\sqrt{-\mu} \left(x - \frac{48ta_0^3}{a_1^3} \right) \right]. \quad (20)$$

In the hyperbolic solution presented in Equation (20), special values were assigned to the constants, and graphs showing the solitary wave at any moment are presented in Figure 5.

Case 6. If

$$\gamma = \frac{4(9a_0^2 + 2\mu a_1^2)}{a_1^2}, \quad a_2 = 0, \quad \beta = \frac{24}{a_1^3}, \quad \alpha = -\frac{72a_0}{a_1^3}, \quad v = -\frac{48a_0^3}{a_1^3}, \quad (21)$$

We obtain the following wave solutions by changing the values of Equation (20) into Equation (9) for Equation (1),

$$u_6(x, t) = a_0 - \sqrt{-\mu} \coth \left[\sqrt{-\mu} \left(x - \frac{48ta_0^3}{a_1^3} \right) \right] a_1. \quad (22)$$

In the hyperbolic solution presented in Equation (22), special values were assigned to the constants, and graphs showing the solitary wave at any moment are presented in Figure 6.

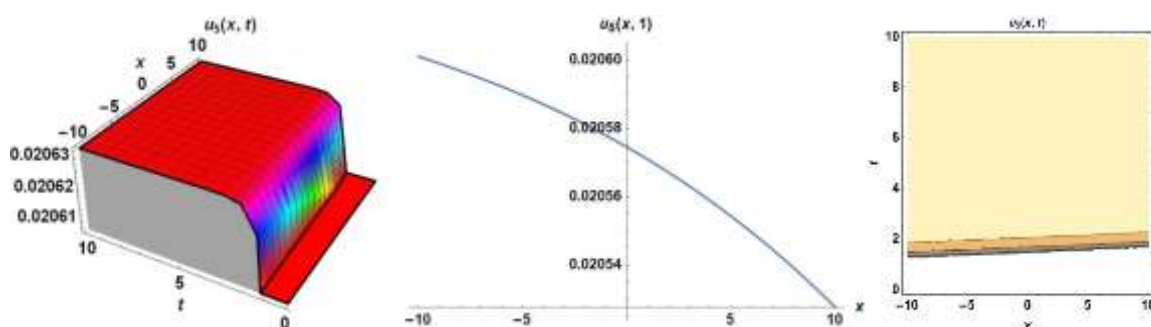


Figure 5. Three-dimensional, two-dimensional, and contour graphs of Equation (20) for $a_0 = 0.02$, $a_1 = 0.02$, $\mu = -0.001$.

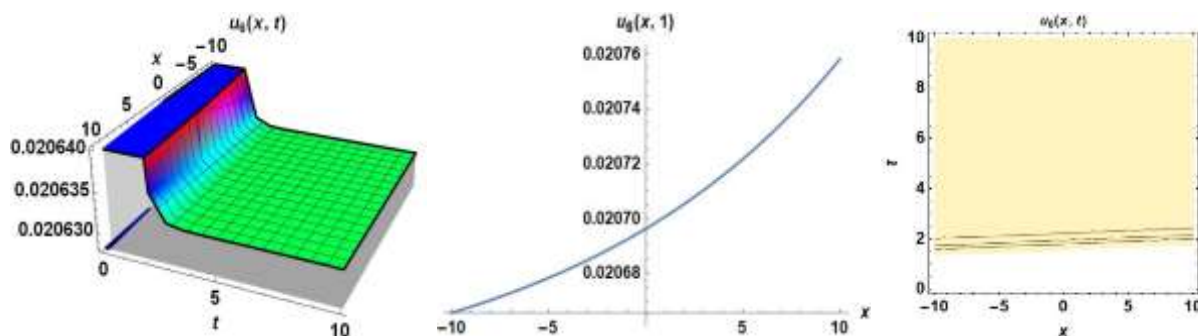


Figure 6. Three-dimensional, two-dimensional, and contour graphs of Equation (22) for $a_0 = 0.02$, $a_1 = 0.02$, $\mu = -0.001$.

Case 7. If

$$\gamma = \frac{4(9a_0^2 + 2\mu a_1^2)}{a_1^2}, \quad a_2 = 0, \quad \beta = \frac{24}{a_1^3}, \quad \alpha = -\frac{72a_0}{a_1^3}, \quad v = -\frac{48a_0^3}{a_1^3}, \quad (23)$$

We obtain the following wave solutions by changing the values of Equation (23) into Equation (9) for Equation (1),

$$u_7(x, t) = a_0 + \sqrt{\mu} a_1 \tan \left[\sqrt{\mu} \left(x - \frac{48ta_0^3}{a_1^3} \right) \right]. \quad (24)$$

In the trigonometric solution presented in Equation (24), special values were assigned to the constants, and graphs showing the solitary wave at any moment are presented in Figure 7.

Case 8. If

$$\gamma = \frac{4(9a_0^2 + 2\mu a_1^2)}{a_1^2}, \quad a_2 = 0, \quad \beta = \frac{24}{a_1^3}, \quad \alpha = -\frac{72a_0}{a_1^3}, \quad v = -\frac{48a_0^3}{a_1^3}, \quad (25)$$

We obtain the following wave solutions by changing the values of Equation (25) into Equation (9) for Equation (1),

$$u_8(x, t) = a_0 - \sqrt{\mu} \cot \left[\sqrt{\mu} \left(x - \frac{48ta_0^3}{a_1^3} \right) \right] a_1. \quad (26)$$

In the trigonometric solution presented in Equation (26), special values were assigned to the constants, and graphs showing the solitary wave at any moment are presented in Figure 8.

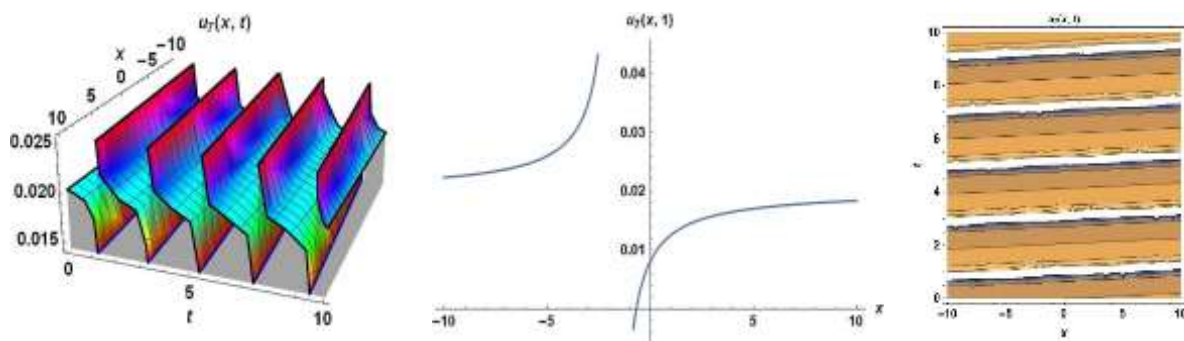


Figure 7. Three-dimensional, two-dimensional, and contour graphs of Equation (24) for $a_0 = 0.02$, $a_1 = 0.02$, $\mu = 0.001$.

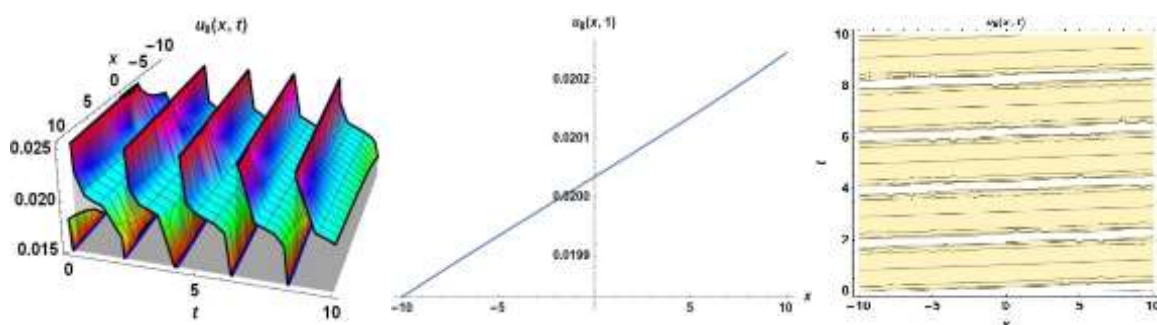


Figure 8. Three-dimensional, two-dimensional, and contour graphs of Equation (26) for $a_0 = 0.02$, $a_1 = 0.02$, $\mu = 0.001$.

CONCLUSION

In this study, the modified sub-equation method was successfully used to construct a set of traveling wave solutions for the K-S equation. As a result, a new set of exact traveling wave solutions was obtained. Three-dimensional, two-dimensional, and contour graphics of the obtained solutions were presented. In addition, Mathematica was used for complex calculations and programming. The results show that the method is dependable and efficient, and can be used to solve different NLEEs. The method's reliability and versatility indicate its potential for wider application to other non-linear differential equations, making it a valuable contribution to the fields of mathematical physics and applied mathematics.

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