STM JOURNALS

Journal of
Polymer & Composites

ISSN: 2321-2810 (Online)
ISSN: 2321-8525 (Print)
Volume 13, Special Issue 1, 2025
DOI (Journal): 10.37591/JoPC

https://journals.stmjournals.com/jopc

Research JoPC

Modelling of The Viscoelastic Composite Curved Panel for The Time Domain Analysis Using TTBDF, $\beta_1 / \beta_2 -$ Bathe and Newmark Method

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Abstract

Viscoelastic materials are extensively used in structures, especially thin-walled structures, for damping. The accurate modelling of the time domain dynamics of the viscoelastic material is essential for appropriately capturing the damping of the viscoelastic material. Various implicit and explicit time integration schemes are available to evaluate time domain response. However, the rightness of the implementation of the time integration scheme to the viscoelastic material model is very essential. In this paper, the non-linear FE model is developed to characterize the time domain behaviour of the composite curved panel. The composite curved panel comprises metal and viscoelastic material. The viscoelastic material is modelled by a four-parameter fractional order derivative model. The time integration method is then coupled with the non-linear FE model and then solved by the Newton-Raphson method to obtain a time domain response. Three different time integration methods such as TTBDF composite method, $\beta_1 / \beta_2 -$ Bathe explicit method and the Newmark implicit method are combined with the non-linear FE model and the respective responses are evaluated. The dynamic response of the composite curved panel is studied under the external harmonic force excitation. The responses of the three given time integration methods are compared. All three-time integration methods accurately capture the forcing frequency with slight variation in the displacement responses due to the implementation of the Grunwald formalism.

Keywords: Four-parameter fractional order derivative model, Newmark implicit method, $\beta_1 / \beta_2 -$ Bathe explicit method, TTBDF composite method

INTRODUCTION

The potential for catastrophic structural failure, particularly in structures subject to resonance or repeated loading cycles, poses a significant risk. Vibrations compromise system performance, increase wear, and generate unwanted noise. These vibrations can also lead to the loss of dynamic stability, making it necessary to introduce damping into the system. To address this issue, viscoelastic materials are commonly used to passively control structure vibrations. By employing the constrained layer damping technique, viscoelastic materials can provide superior damping performance [1,2]. In this study, a curved panel comprised of a viscoelastic core sandwiched between metal panels was examined. Viscoelastic materials exhibit frequency-dependent dynamic properties, necessitating proper modelling of these materials. Classical linear viscoelastic models face challenges to characterize material behaviour as it involves too many variables. In this issue, fractional derivative operators which considers strain and stress

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Received Date: May 17, 2024
Accepted Date: June 15, 2024
Published Date: December 06, 2024

Citation: Vishwanil Sarnaik, Mekala Anil, Satyajit Panda, S. Kanaragaj. Modelling of the viscoelastic composite curved panel for the time domain analysis using TTBDF, β_1/β_2 −Bathe and Newmark method. Journal of Polymer & Composites. 2025; 13(Special Issue 1): S204–S222p.

variables are a good alternative choice. The fractional order model was first theoretically conceptualized by Bagley and Torvik [3]. with the physical interpretation of the model in thermo-dynamic problems.

The fractional order model was initially limited to the curve fitting of experimental data. However, researchers are now focusing on integrating the fractional order derivative constitutive equation into finite element (FE) formulations. To calculate the fractional-order derivative in the material model for the time domain response of structures consisting of viscoelastic material using the finite element model, Grunwald formalism [4] is employed along with a time integration scheme [5-8]. It is very important to choose the right kind of time integration scheme to accurately predict the time domain response of the structure. The time domain responses can be obtained by using implicit or explicit schemes. But the preferred scheme for nonlinear problems is implicit schemes as large time steps can be used without comprising the accuracy of the solution. However, to solve the non-linear problems with implicit integration scheme, an iterative procedure such as newton raphson scheme are widely used within each time step in finite element analysis. Explicit schemes can also be coupled with the newton raphson method. The correct implementation of the algorithm is crucial to avoid theoretically incompatible or unstable solutions. Also, some time integrations schemes have multiple sub steps. Thus, the consideration of the mathematical characteristics of the time integration schemes while coupling with the viscoelastic material model is utmost important to obtain the nonlinear response.

This work demonstrates a non-linear Fe formulation for the time domain response of viscoelastic composite curved panel by incorporating three different time integration schemes, namely the Newmark implicit method [9]. β_1/β_2- Bathe explicit [10], and TTBDF composite method [11]. The viscoelastic material is described using a fractional order derivative model with four-parameters. The frequency-dependent properties of the viscoelastic material are obtained experimentally, and four parameters are determined based on the experimental data. The evaluation of the non-linear responses is accomplished through the development of an algorithm for each time integration scheme. The finite element model is verified through the reference problem, and the responses evaluated by the Newmark implicit method, β_1/β_2-Bathe explicit method, and TTBDF composite method for harmonic force excitation are compared. The effectiveness of the three different time integration schemes to appropriately and accurately capture the forcing frequency and amplitude response of the viscoelastic curved panel is observed.

CONSTITUTIVE RELATIONS

A fractional order derivative model with four parameters is used to describe the viscoelastic material behaviour. The constitutive relation for two-dimensional plain strain condition is given as follows [12]:

$$\sigma+\tau^{\alpha}\frac{d^{\alpha}\sigma}{dt^{\alpha}}=C\varepsilon+\tau^{\alpha}\frac{E_{\infty}}{E_0}C\frac{d^{\alpha}\varepsilon}{dt^{\alpha}} \tag{1}$$

$$C=\frac{E_0}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

where σ,ε and C are stress vector, strain vector and constitutive matrix, respectively. The four parameters of the model E_0, E_{∞}, α and τ are called as relaxed modulus, instantaneous modulus, fractional order and relaxation time, respectively; ν is the Poisson’s ratio. The stress and strain vector for 2-D plain strain are given as follows:

$$\sigma=\begin{bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{bmatrix},\quad \varepsilon=\begin{bmatrix}\varepsilon_x\\ \varepsilon_z\\ \gamma_{xz}\end{bmatrix} \tag{2}$$

Further, to get rid of the fractional operator ,anelastic strain vector is introduced in the Eq. (1). The Eq(1) can be further written as

$$\sigma=\frac{E_{\infty}}{E_0}C\left(\varepsilon-\bar{\varepsilon}\right) \tag{3}$$

Next, substituting the Eq.(3) in Eq.(1) to obtain Eq.(4), anelastic strain ($\bar{\varepsilon}$) can be represented as follows:

$$\bar{\varepsilon}+\tau^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}\bar{\varepsilon}=\frac{E_{\infty}-E_0}{E_0}\varepsilon \tag{4}$$

Now, the fractional operator $\left(d^{\alpha}\right)$ which is present in Eq.(4) can be eliminated by using the Grunwald definition, (Schmidt and Gaul) [4] and is given as follows:

$$D^{\alpha}g(t)=\frac{1}{\Delta t}\sum_{k=1}^{S_t}B_{k+1}g\left\{t-\left(k\Delta t\right)\right\} \tag{5}$$

Based on Grunwald definition, at the $\left(s+1\right)th$ time-step ,the anelastic strain vector can be written as

$$\bar{\varepsilon}_{s+1}=\left(\frac{E_{\infty}-E_0}{E_0}\right)\left(1-c\right)\varepsilon_{s+1}\text{ - }c\left\{\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k}\right\}\; B_{k+1}=\frac{k-\left(\alpha+1\right)}{k}B_k,\quad B_1=1,\quad c=\frac{\tau^{\alpha}}{\tau^{\alpha}+\left(\Delta t\right)^{\alpha}} \tag{6}$$

Where, $\left(\Delta t\right)$, B_{k+1} and $\left(S_t\right)$ are equal time steps, Grunwald constant and number of time divisions. Similarly, stress vector can be produced by substituting the anelastic strain vector in Eq.(3).

$$\sigma_{s+1}^{q}=C_1^{q}\varepsilon_{s+1}+C_2^{q}\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k},\quad q=1\; C_1^{q}=\left\langle\frac{E_0-c\left(E_{\infty}-E_0\right)}{E_0}\right\rangle C,\quad C_2^{q}=\left(\frac{cE_{\infty}}{E_0}\right)C, \tag{7}$$

Now, at the $\left(s+1\right)th$ time-step, isotropic elastic material's constitutive relation can be derived by substituting $c=0$ in Eq.(7). Here, q with the value of 1 and 2 represents the viscoelastic material and elastic isotropic material, respectively (Figure 1)

$$\sigma_{s+1}^{q}=C\varepsilon_{s+1},\quad q=2 \tag{8}$$

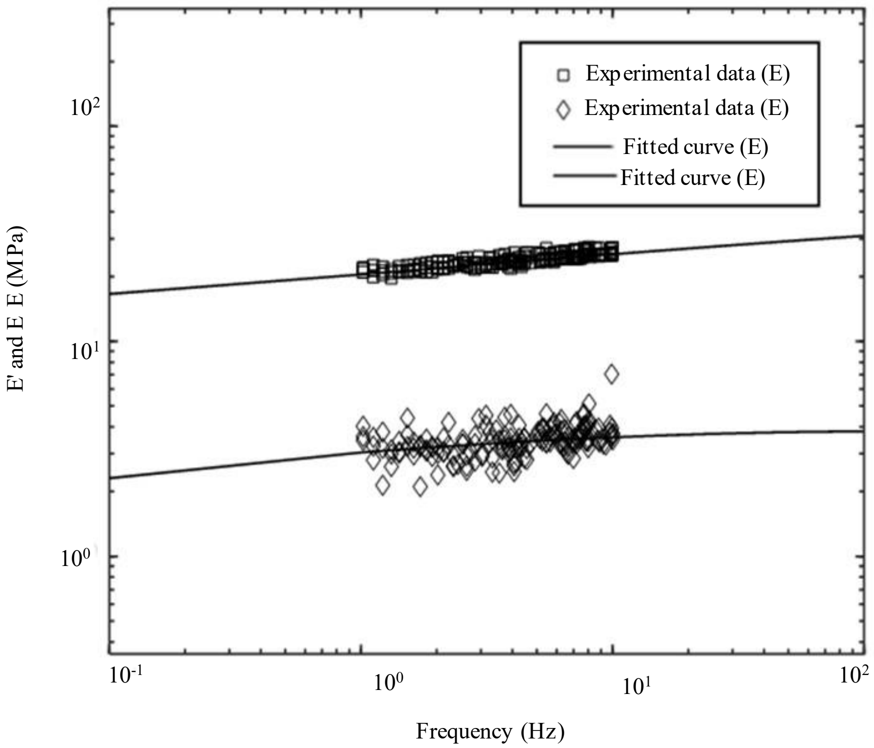


Figure 1. Storage modulus ( ) and loss modulus ( ) of the nitrile rubber obtained through DMA and the corresponding fitted functional (Eq.(9)) curves.

Next, the four parameters present in the fractional model is unique for each material. In order to find the four parameters, initially, the dynamic mechanical analysis is performed. The dynamic mechanical analysis gives the frequency dependent storage and loss modulus. The relation for storage and loss modulus for four parameter fractional model [13] is given as follows:

$$E^*(\omega) = E'(\omega) + iE''(\omega), \quad i = \sqrt{-1} \tag{9}$$
$$E'(\omega) = \frac{E_o + (E_\infty + E_o)(\omega\tau)^\alpha \cos(\pi\alpha/2) + E_\infty(\omega\tau)^{2\alpha}}{1 + 2(\omega\tau)^\alpha \cos(\pi\alpha/2) + (\omega\tau)^{2\alpha}}$$
$$E''(\omega) = \frac{(E_\infty - E_o)(\omega\tau)^\alpha \sin(\pi\alpha/2)}{1 + 2(\omega\tau)^\alpha \cos(\pi\alpha/2) + (\omega\tau)^{2\alpha}}$$

The relation for storage and loss modulus are then used to obtain the four parameters by fitting the curves into the experimental data obtained by dynamic mechanical analysis. In this paper, nitrile rubber is used as viscoelastic material and aluminum as isotropic material. The four parameters for the nitrile rubber are $E_0 = 9.21 Mpa,$ $\alpha = 0.1924$, $\tau = 5.198 \times 10^{-6}$ and $E_\infty = 104.2e6 MPa.$ Also, the density and poison’s ratio of the nitrile rubber are $\rho = 1163 Kg/m^3$ and $\upsilon = 0.49$.

FE FORMULATION

The curved panel with viscoelastic core sandwiched between the metal layers is considered. The panel is fixed at both ends with line force acting at the top of the panel throughout the length (b) in y-direction. Also, the mass (m_l) is attached along the length in in y-direction at the top of the panel. The panel is considered in plain strain condition in $x-z$ plane considering the boundary and loading condition. The inner and outer panels are made of metal described by radii (R_1, R_2) and radii (R_3, R_4) ,respectively. The viscoelastic core is described by radii (R_2, R_3). The thickness of the core and metal panels are h_1 and h_2 ,respectively. The panel' schematic diagram in x-z plane is shown in .

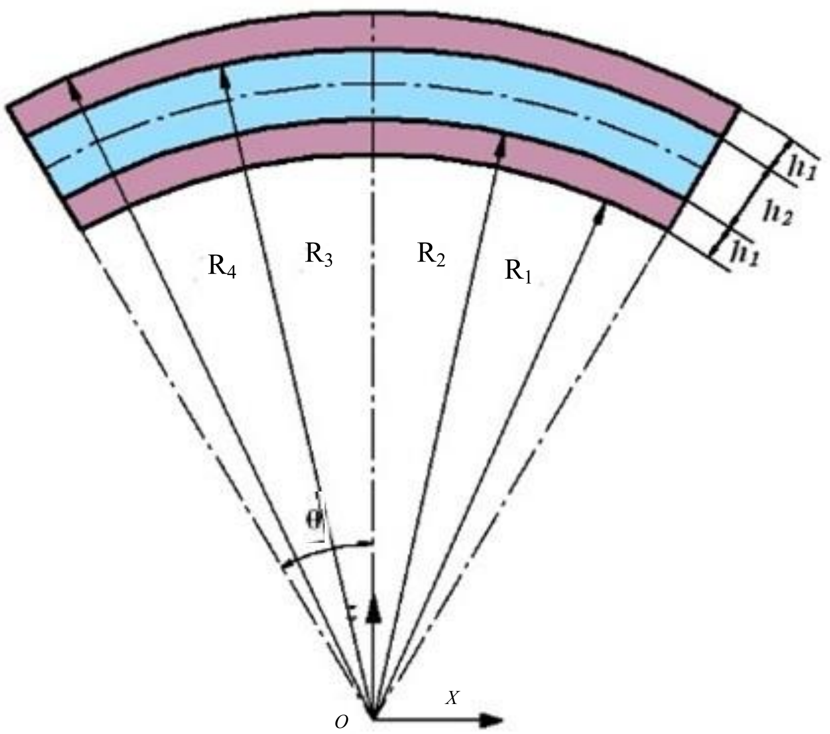


Figure 2. Geometric dimensions of the curved panel.

The displacement (u, w) at any point in the x-z plane is written as a vector and given by Eq.(10).

$$V = \begin{bmatrix} u \\ w \end{bmatrix} \tag{10}$$

Now, the strain vector (ε) in terms of the displacement vector components is written as follows:

$$\varepsilon = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial w}{\partial z}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 \\ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \end{bmatrix} \tag{11}$$

Next, to derive the Fe formulation, initially, the total potential energy $(\delta\Pi_p)$ and total kinetic energy $(\delta\Pi_k)$ are considered in the first variational form.

$$\delta\Pi_p = \sum_{q=1}^{2}\int_{A^q} (\delta\varepsilon_{s+1})' \left\{ C_1^q \varepsilon_{s+1} + C_2^q \sum_{k=1}^{S_t} B_{k+1}\bar{\varepsilon}_{s+1-k} \right\} bdA^q - (\delta w_{s+1})' \left\{ F^i + F^c \sin(\omega t) \right\}$$

$$\delta\Pi_k = \sum_{q=1}^{2}\int_{A^q} \left\{ (\delta\dot{V}_{s+1})' \rho^q \dot{V}_{s+1} \right\} bdA^q + (\delta\dot{w}_{s+1})' \left(\frac{m_l}{b}\right) \dot{w}_{s+1}|_{(x_t, z_t)} \tag{12a)(12b}$$

In Eqs.((12a)-(12b)), the area and the density of the q^{th} material is denoted by the A^q and ρ^q. Also, F^i, F^c and ω are the constant force, cosine force amplitude and the angular frequency, respectively. Next, the governing equation can be found by initially deriving the overall first variational form by using Hamilton principle which is represented by Eq.(13).

$$\int_{t_1}^{t_2} (\delta T_p - \delta T_k) dt = 0 \tag{13}$$

Now, using the Eqs.((12a) -(12b)) and (Eq.(13)), the overall first variational form is given by Eq.(14)

$$\sum_{q=1}^{2}\int_{A^q} \begin{bmatrix} (\delta V_{s+1})^T \rho^q \ddot{V}_{s+1} + (\delta\varepsilon_{s+1})' C_1^q \varepsilon_{s+1} \\ + \left\{ (\delta\varepsilon_{s+1})' C_2^q \sum_{k=1}^{S_t} B_{k+1}\bar{\varepsilon}_{s+1-k} \right\} \end{bmatrix} bdA^q + (\delta w_{s+1})' \left(\frac{m_l}{b}\right) \ddot{w}_{s+1}|_{(x_t, z_t)} - (\delta w_{s+1})' \left\{ F^i + F^c \sin(\omega t) \right\} = 0 \tag{14}$$

Next, the above non-linear equation can be solved by newton Raphson method considering the overall displacement as a sum of the incremental displacement (ΔV) and the reference displacement $({}^iV)$.

$$V = ({}^iV + \Delta V) \quad {}^iV = \left\{ {}^iu \quad {}^iw \right\}^T, \quad \Delta V = \left\{ \Delta u \quad \Delta w \right\}^T \tag{15}$$

Now, the reference strain vectors $({}^i\varepsilon_l, {}^i\varepsilon_{n1}, {}^i\varepsilon_{n2})$ and incremental strain vectors $(\Delta\varepsilon_l, \Delta\varepsilon_{n1}, \Delta\varepsilon_{n2}, \Delta\varepsilon_{nn1}, \Delta\varepsilon_{nn2})$ corresponding to the reference displacement $({}^iV)$ and incremental displacement (ΔV) are defined as Eq.(16) and given in Appendix.

$$\varepsilon = {}^i\varepsilon + \Delta\varepsilon, \quad {}^i\varepsilon = {}^i\varepsilon_l + {}^i\varepsilon_{n1} + {}^i\varepsilon_{n2}, \quad \Delta\varepsilon = \Delta\varepsilon_l + \Delta\varepsilon_{n1} + \Delta\varepsilon_{n2} + \Delta\varepsilon_{nn1} + \Delta\varepsilon_{nn2} \tag{16}$$

Next, the overall first variational form in terms of the reference and incremental strain vectors can be rewritten in the new form by substituting Eqs.((15)-(16)) in Eq.(14) as

$$\delta\Pi_{\mathrm{t}}+\delta\Pi_{k}+\delta\Pi_{\mathrm{i}}+\delta\Pi_{v}-\delta\Pi_{e}=0 \tag{17a}$$

$$\delta\Pi_{\mathrm{i}}=\sum_{q=1}^{2}\int_{A^q}\begin{bmatrix}\left(\delta\Delta\varepsilon_l\right)' C_1^q\,{}^i\varepsilon_l+\left(\delta\Delta\varepsilon_l\right)' C_1^q\,{}^i\varepsilon_{n1}+\left(\delta\Delta\varepsilon_l\right)' C_1^q\,{}^i\varepsilon_{n2}+\\ \left(\delta\Delta\varepsilon_{nn1}\right)' C_1^q\,{}^i\varepsilon_l+\left(\delta\Delta\varepsilon_{nn1}\right)' C_1^q\,{}^i\varepsilon_{n1}+\left(\delta\Delta\varepsilon_{nn1}\right)' C_1^q\,{}^i\varepsilon_{n2}+\\ \left(\delta\Delta\varepsilon_{nn2}\right)' C_1^q\,{}^i\varepsilon_l+\left(\delta\Delta\varepsilon_{nn2}\right)' C_1^q\,{}^i\varepsilon_{n1}+\left(\delta\Delta\varepsilon_{nn2}\right)' C_1^q\,{}^i\varepsilon_{n2}\end{bmatrix}bdA^q \tag{17b}$$

$$\delta\Pi_k=\sum_{q=1}^{2}\int_{A^q}\left[\left(\delta\Delta V_{s+1}\right)'\rho^q\left({}^i\ddot{V}_{s+1}+\Delta\ddot{V}_{s+1}\right)\right]bdA^q+\left(\delta\Delta w_{s+1}\right)'\left(\frac{m_l}{b}\right)\left({}^i\ddot{w}_{s+1}+\Delta\ddot{w}_{s+1}\right)\Big|_{(x_t,z_t)} \tag{17c}$$

$$\delta\Pi_v=\sum_{q=1}^{2}\int_{A^q}\left[\left(\delta\Delta\varepsilon_l+\delta\Delta\varepsilon_{nn1}+\delta\Delta\varepsilon_{nn2}\right)'\left\{C_2^q\right\}\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k}\right]bdA^q \tag{17d}$$

$$\delta\Pi_{\mathrm{t}}=\sum_{q=1}^{2}\int_{A^q}\begin{bmatrix}\left(\delta\Delta\varepsilon_l\right)' C_1^q\Delta\varepsilon_l+\left(\delta\Delta\varepsilon_l\right)' C_1^q\Delta\varepsilon_{nn1}+\left(\delta\Delta\varepsilon_l\right)' C_1^q\Delta\varepsilon_{nn2}\\ +\left(\delta\Delta\varepsilon_{nn1}\right)' C_1^q\Delta\varepsilon_l+\left(\delta\Delta\varepsilon_{nn1}\right)' C_1^q\Delta\varepsilon_{nn1}+\left(\delta\Delta\varepsilon_{nn1}\right)' C_1^q\Delta\varepsilon_{nn2}\\ +\left(\delta\Delta\varepsilon_{nn2}\right)' C_1^q\Delta\varepsilon_l+\left(\delta\Delta\varepsilon_{nn2}\right)' C_1^q\Delta\varepsilon_{nn1}+\left(\delta\Delta\varepsilon_{nn2}\right)' C_1^q\Delta\varepsilon_{nn2}\end{bmatrix}bdA^q$$
$$+\sum_{q=1}^{2}\int_{A^q}\left[\left(\delta\Delta\varepsilon_{n1}+\delta\Delta\varepsilon_{n2}\right)'\left\{C_1^q\left({}^i\varepsilon_l+{}^i\varepsilon_{n1}+{}^i\varepsilon_{n2}\right)+C_2^q\left(\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k}\right)\right\}\right]bdA^q \tag{17e}$$

$$\delta\Pi_e=\left(\delta\Delta w_{s+1}\right)'\left\{F^i+F^c\cos(\omega t)\right\} \tag{17f}$$

urther, for the derivation of FE model, the curved panel is discretized by nine node elements in $x-z$ plane. Also, the displacement vectors $\left({}^iV,\Delta V\right)$ in terms of shape function $\left(N_{sp}\right)$ and elemental displacement vector $\left({}^iV_e,\Delta V_e\right)$ an any point in the element can be expressed as follows:

$${}^iV=N_{sp}\,{}^iV_e,\quad \Delta V=N_{sp}\Delta V_e\ \ N_{sp}=\begin{bmatrix}N_j & 0\\ 0 & N_j\end{bmatrix}\qquad j=(1,2,3.......9) \tag{18}$$

The reference strain vectors $\left({}^i\varepsilon_l,{}^i\varepsilon_{n1},{}^i\varepsilon_{n2}\right)$ and incremental strain vectors $\left(\Delta\varepsilon_l,\Delta\varepsilon_{nn1},\Delta\varepsilon_{nn2}\right)$ in discretized form are illustrated by Eq.(19).

$${}^i\varepsilon_l=B_l\,{}^iV_e,\quad {}^i\varepsilon_{n1}=B_{n1}\,{}^iV_{eP}\,{}^iV_e,\quad {}^i\varepsilon_{n2}=B_{n2}\,{}^iV_{eP}\,{}^iV_e \tag{19}$$
$$\Delta\varepsilon_l=B_l\Delta V_e,\quad \Delta\varepsilon_{nn1}=B_{nn1}\,{}^iV_{eP}\Delta V_e,\quad \Delta\varepsilon_{nn2}=B_{nn2}\,{}^iV_{eP}\Delta V_e\ \ {}^iV_{eP}=I\otimes{}^iV_e$$

I is identity matrix of order 18. Now, the variational form due to internal energy $(\delta\Pi_{\mathrm{i}})$ (Eq.(17b)) can be written again by substituting Eqs((18)-(19)) in the Eq.(17b) as Eq.(20)

$$\delta\Pi_{\mathrm{i}}=\left(\delta\Delta V_e\right)'_{s+1}\begin{Bmatrix}K_{ll}^e+K_{\ln 1}\,{}^iV_{eP}+K_{\ln 2}^e\,{}^iV_{eP}+\left({}^iV_{eP}\right)' K_{nn1l}^e+\\ \left({}^iV_{eP}\right)' K_{nn1n1}^e\,{}^iV_{eP}+\left({}^iV_{eP}\right)' K_{nn1n2}^e\,{}^iV_{eP}+\left({}^iV_{eP}\right)' K_{nn2l}^e\\ +\left({}^iV_{eP}\right)' K_{nn2n1}^e\,{}^iV_{eP}+\left({}^iV_{eP}\right)' K_{nn2n2}^e\,{}^iV_{eP}\end{Bmatrix} \tag{20}$$

Journal of Polymer & Composites
Volume 13, Special Issue 1
ISSN: 2321-2810 (Online), ISSN: 2321-8525 (Print)

Where,

$$K_{ll}^{e}=\int_{A_e}(B_l)^{'}C_1^eB_l\,dA_e \qquad K_{\ln1}^{e}=\int_{A_e}(B_l)^{'}C_1^eB_{n1}\,dA_e \qquad K_{\ln2}^{e}=\int_{A_e}(B_l)^{'}C_1^eB_{n2}\,dA_e$$

$$K_{nn1l}^{e}=\int_{A_e}(B_{nn1})^{'}C_1^eB_l\,dA_e \quad K_{nn1n1}^{e}=\int_{A_e}(B_{nn1})^{'}C_1^eB_{n1}\,dA_e \quad K_{nn1n2}^{e}=\int_{A_e}(B_{nn1})^{'}C_1^eB_{n2}\,dA_e$$

$$K_{nn1n2}^{e}=\int_{A_e}(B_{nn1})^{'}C_1^eB_{n2}\,dA_e \quad K_{nn2n1}^{e}=\int_{A_e}(B_{nn2})^{'}C_1^eB_{n1}\,dA_e \quad K_{nn2n2}^{e}=\int_{A_e}(B_{nn2})^{'}C_1^eB_{n2}\,dA_e$$

Next, the Eq.(21) represents the kinetic energy expression $(\delta\Pi_k)$ which can be obtained by substituting Eq.(18) in Eq.(17c). Also, external mass's location matrix is $(\mathrm{Im})^e$ as shown in Eq.(21)

$$\delta\Pi_k=(\delta\Delta V_e)_{s+1}^{'}\{M^e\}({}^{i}\ddot{V}_e)_{s+1}+(\delta\Delta V_e)_{s+1}^{'}\{M^e\}(\Delta\ddot{V}_e)_{s+1} \tag{21}$$

$$M^e=m^e+(\mathrm{Im})^e\left(\frac{m_l}{b}\right),\quad m^e=\int_{A^e}\left(N_{sp}{}^{T}p^eN_{sp}\right)b\,dA_e$$

Further, substituting Eq. (19) in Eq.(17d), the dissipative energy variational form $(\delta\Pi_v)$ can be represented in the form as given by Eq.(22).

$$\delta\Pi_v=K_{visco}^{e}\{C_2\}\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k}^{e} \tag{22}$$

$$K_{visco}^{e}=(B_l)^{'}+({}^{i}V_{eP})^{'}(B_{nn1})^{'}+({}^{i}V_{eP})^{'}(B_{nn2})^{'}$$

$$\left(F_{visco}^{e}\right)_{s+1}=K_{visco}^{e}\{C_2\}\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k}^{e}$$

The dissipative energy term $(\delta\Pi_v)$ consists of the summation of the anelastic strain history and the anelastic strain at $(s+1)th$ time step is evaluated by Eq.(6). $\left(F_{visco}^{e}\right)_{s+1}$ is the damping force vector. Also, the new expression given by Eq.(23) for the variational form of the external work $(\delta\Pi_e)$ is obtained by substituting Eq.(18) in Eq.(17f). In Eq.(23), $(P_l)^e$ is the elemental location matrix for the force vector.

$$\delta\Pi_e=(\delta\Delta V_e)_{s+1}^{'}(P_l)^e(F_{s+1}^{e})\;\;F_{s+1}^{e}=\left\{F^i+F^c\sin(\omega t)\right\} \tag{23}$$

Next, the overall tangent stiffness's variational form $(\delta\Pi_{\mathrm{t}})$ is obtained by substituting Eq.(18) and (19) in Eq.(17e) and is illustrated as

$$\delta\Pi_{\mathrm{t}}=(\delta\Delta V_e)_{s+1}^{'}\left\{\begin{array}{l}\Delta K_{ll}^{e}+\Delta K_{\ln n1}^{e}\,{}^{i}V_{eP}+\Delta K_{\ln n2}^{e}\,{}^{i}V_{eP}+({}^{i}V_{eP})^{'}\Delta K_{nn1l}^{e}\\+({}^{i}V_{eP})^{'}\Delta K_{nn1nn1}^{e}\,{}^{i}V_{eI}+({}^{i}V_{eP})^{'}\Delta K_{nn1nn2}^{e}\,{}^{i}V_{eP}+({}^{i}V_{eP})^{'}\Delta K_{nn2l}^{e}\\+({}^{i}V_{eP})^{'}\Delta K_{nn2nn1}^{e}\,{}^{i}V_{eI}+({}^{i}V_{eP})^{'}\Delta K_{nn2nn2}^{e}\,{}^{i}V_{eP}+\Delta K_{g}^{e}\end{array}\right\} \tag{24}$$

$$\Delta K_{ll}^{e}=\int_{A_e}\left(B_l\right)^{'}C_1^eB_ldA_e \qquad \Delta K_{\ln n1}^{e}=\int_{A_e}\left(B_l\right)^{'}C_1^eB_{nn1}dA_e \qquad \Delta K_{\ln n2}^{e}=\int_{A_e}\left(B_l\right)^{'}C_1^eB_{nn2}dA_e$$

$$\Delta K_{nn1l}^{e}=\int_{A_e}\left(B_{nn1}\right)^{'}C_1^eB_ldA_e \quad \Delta K_{nn1nn1}^{e}=\int_{A_e}\left(B_{nn1}\right)^{'}C_1^eB_{nn1}dA_e \quad \Delta K_{nn1nn2}^{e}=\int_{A_e}\left(B_{nn1}\right)^{'}C_1^eB_{nn2}dA_e$$

$$\Delta K_{nn2l}^{e}=\int_{A_e}\left(B_{nn2}\right)^{'}C_1^eB_ldA_e \quad \Delta K_{nn2nn1}^{e}=\int_{A_e}\left(B_{nn2}\right)^{'}C_1^eB_{nn1}dA_e \quad \Delta K_{nn2nn2}^{e}=\int_{A_e}\left(B_{nn2}\right)^{'}C_1^eB_{nn2}dA_e$$

$$\Delta K_{g}^{e}=\int_{A_e}\left(B_g\right)^{'}S_mB_gdA_e$$

Further, the overall tangent stiffness matrix consisting the geometric stiffness matrix $\left(\Delta K_g^e\right)$ includes anelastic strain history responsible for damping force vector $\left(F_{visco}^{e}\right)_{s+1}$. Further, the elemental governing equation of motion is obtained by substituting Eqs.(20)-(24) in Eq.(17a) and the governing equation is given by Eq.(25).

$$M^e\left(\Delta\ddot{V}_e\right)_{s+1}+K_T^e\left({}^{i}V_e\right)_{s+1}=R^e \tag{25}$$

Where,

$$R^e=F_{s+1}^e-M^e\ddot{V}_{s+1}^e-K^eV_{s+1}^e-\left(F_{visco}^e\right)_{s+1} \quad M^e=m^e+\left(\mathrm{Im}\right)^e\left(\frac{m_l}{b}\right) \quad \left(F_{visco}^e\right)_{s+1}=K_{visco}^e\left\{C_2^e\right\}\sum_{k=1}^{S_t}B_{k+1}\bar{\varepsilon}_{s+1-k}^e$$

$$F_{s+1}^e=\left(P_l\right)^e\left\{F^i+F^c\sin(\omega t)\right\}$$

$$K^e=\begin{Bmatrix} K_{ll}^e+K_{\ln 1}^e\,{}^{i}V_{eP}+K_{\ln 2}^e\,{}^{i}V_{eP}+\left({}^{i}V_{eP}\right)^{'}K_{nn1l}^e+\left({}^{i}V_{eP}\right)^{'}K_{nn1n1}^e\,{}^{i}V_{eP}+ \\ \left({}^{i}V_{eP}\right)^{'}K_{nn1n2}^e\,{}^{i}V_{eP}+\left({}^{i}V_{eP}\right)^{'}K_{nn2l}^e+\left({}^{i}V_{eP}\right)^{'}K_{nn2n1}^e\,{}^{i}V_{eP}+\left({}^{i}V_{eP}\right)^{'}K_{nn2n2}^e\,{}^{i}V_{eP} \end{Bmatrix}$$

$$K_T^e=\begin{Bmatrix} \Delta K_{ll}^e+\Delta K_{\ln n1}^e\,{}^{i}V_{eP}+\Delta K_{\ln n2}^e\,{}^{i}V_{eP}+\left({}^{i}V_{eP}\right)^{'}\Delta K_{nn1l}^e+\left({}^{i}V_{eP}\right)^{'}\Delta K_{nn1nn1}^e\,{}^{i}V_{eP}+ \\ \left({}^{i}V_{eP}\right)^{'}\Delta K_{nn1nn2}^e\,{}^{i}V_{eP}+\left({}^{i}V_{eP}\right)^{'}\Delta K_{nn2l}^e+\left({}^{i}V_{eP}\right)^{'}\Delta K_{nn2nn1}^e\,{}^{i}V_{eP}+ \\ \left({}^{i}V_{ep}\right)^{'}\Delta K_{nn2nn2}^e\,{}^{i}V_{eP}+\Delta K_g^e \end{Bmatrix}$$

Next, the elemental matrices are assembled to obtain the global governing equation of motion and are given as follows.

$$M_g\Delta\ddot{V}+K_{gT}\Delta V=R_g \tag{26}$$

Where,

$$R_g = \left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}\; M_g = m + \left(\mathrm{Im}\right)\left(\frac{m_l}{b}\right)$$

$$K_g = \left\{\begin{matrix} K_{ll} + K_{\ln 1}\,{}^{i}V_P + K_{\ln 2}\,{}^{i}V_p + \left({}^{i}V_P\right)' K_{nn1l} + \left({}^{i}V_P\right)' K_{nn1n1}\,{}^{i}V_P + \left({}^{i}V_P\right)' K_{nn1n2}\,{}^{i}V_P + \\ \left({}^{i}V_P\right)' K_{nn2l} + \left({}^{i}V_P\right)' K_{nn2n1}\,{}^{i}V_P + \left({}^{i}V_P\right)' K_{nn2n2}\,{}^{i}V_P \end{matrix}\right\}$$

$$K_{gT} = \left\{\begin{matrix} \Delta K_{ll} + \Delta K_{\ln n1}\,{}^{i}V_P + \Delta K_{\ln n2}\,{}^{i}V_P + \left({}^{i}V_P\right)' \Delta K_{nn1l} + \left({}^{i}V_P\right)' \Delta K_{nn1nn1}\,{}^{i}V_P + \\ \left({}^{i}V_P\right)' \Delta K_{nn1nn2}\,{}^{i}V_p + \left({}^{i}V_P\right)' \Delta K_{nn2l} + \left({}^{i}V_P\right)' \Delta K_{nn2nn1}\,{}^{i}V_P + \\ \left({}^{i}V_P\right)' \Delta K_{nn2nn2}\,{}^{i}V_P + \Delta K_g \end{matrix}\right\}$$

$$\left(F_{visco}\right)_{s+1} = K_{visco}\left\{C_2^q\right\}\sum_{k=1}^{S_t} B_{k+1}\bar{\varepsilon}_{s+1-k}, q = 2\; \left(F_g\right)_{s+1} = \left(P_l\right)\left\{F^i + F^c\cos(\omega t)\right\}$$

IMPLEMENTATION OF TIME INTEGRATION METHODS

There are many methods to solve the time domain problem. In this paper, three different types of time integration methods are discussed such as Newmark method, β_1/β_2-Bathe method and, the TTBDF method. All mentioned time integration methods are integrated into the FE model defined by Eq.((26)). The algorithm for all given three-time integration schemes is explained in detail in the below subsections.

Newmark Method

The Newmark method is the most common and widely used method. The Newmark method relates the displacement, velocity and acceleration at the time $(s+1)\Delta t$ with displacement, velocity and acceleration at the time $(s)\Delta t$. The velocity and displacement relations are given as follows.

$$\dot{V}_{s+1} = \dot{V}_s + (1-\delta)(\Delta t)\ddot{V}_s + (\delta\Delta t)\ddot{V}_{s+1}\; V_{s+1} = V_s + (\Delta t)\dot{V}_s + (0.5-\alpha_n)(\Delta t)^2\ddot{V}_s + (\alpha_n)(\Delta t)^2\ddot{V}_{s+1} \tag{27}$$

Where, $V_{s+1}, \dot{V}_{s+1}$ and $\ddot{V}_{s+1}$ are the displacement, velocity and acceleration vectors at time t_{s+1}; α_n and δ are parameters of the Newmark time integration scheme. In order to solve the equation of motion (Eq.(26)), the Eq.(27) is substituted in Eq.(26) to obtain the equation of motion in terms of displacement vectors $\left({}^{i}V_{s+1}, \Delta V_{s+1}\right)$.

$$\Delta V_{s+1} = \left(a_0 M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}\right\} \tag{28}$$

Algorithm implementation

The Newmark method along with the newton Raphson method is used here because of its extensibility in terms of structural dynamics application. The time domain responses of curved panel is obtained by implementing the following algorithm.

1. Required Data
(a) Variables for integration: δ, α_n
(b) Material properties: E_0, E_∞, α, τ
(c) Assembled matrices: K_g, M_g at t=0

2. At the beginning at t=0

$$ {}^{i}V_0, {}^{i}\dot{V}_0 = 0, {}^{i}\ddot{V}_0 = M_g^{-1}\left\{\left(F_g\right)_0 - K_g\,{}^{i}V_0\right\}, \bar{\varepsilon}_0 = (1-c)\left(\frac{E_\infty - E_0}{E_\infty}\right)\varepsilon_0 $$

Where,

${}^{i}V_0 = 0, {}^{i}\dot{V}_0 = 0$, ${}^{i}V_0, {}^{i}\dot{V}_0, {}^{i}\ddot{V}_0, \bar{\varepsilon}_0$ is initial displacement, velocity, acceleration and anelastic strain, respectively.

3. Enter the time step loop, consider t_{s+1} at this step, we know following parameters: ${}^{i}V_s, {}^{i}\dot{V}_s, {}^{i}\ddot{V}_s, \bar{\varepsilon}_s$

4. Calculate the residual load vector R_g with ${}^{i}V_{s+1} = {}^{i}V_s$ as initial guess and calculate

$$ {}^{i}\dot{V}_{s+1}, {}^{i}\ddot{V}_{s+1}, R_g = \left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1} $$

5. While $Norm(R_g) > TOL$

a. Calculate the incremental displacement:

$$ \Delta V_{s+1} = \left(a_0 M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}\right\} $$

b. Update The Displacement

$$ V_{s+1} = {}^{i}V_{s+1} + \Delta V_{s+1} $$

c. Update the velocity and acceleration

$$ {}^{i}\dot{V}_{s+1} = a_1({}^{i}V_{s+1} - {}^{i}V_s) - a_4\,{}^{i}\dot{V}_s - a_5\,{}^{i}\ddot{V}_s $$

$$ {}^{i}\ddot{V}_{s+1} = a_0({}^{i}V_{s+1} - {}^{i}V_s) - a_2\,{}^{i}\dot{V}_s - a_3\,{}^{i}\ddot{V}_s $$

Where,

$$ a_0 = \left(\frac{1}{\alpha_n\left(\Delta t\right)^2}\right), a_1 = \left(\frac{\delta}{\alpha_n\left(\Delta t\right)}\right), a_2 = \left(\frac{1}{\alpha_n\left(\Delta t\right)}\right), a_2 = \left(\frac{1}{\alpha_n\left(\Delta t\right)}\right) $$

$$ a_4 = \left(\frac{\delta}{\alpha_n}\right) - 1, a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha_n} - 2\right), a_6 = \Delta t\left(1-\delta\right), a_6 = \Delta t\left(1-\delta\right) $$

d. Again, calculate the residual R_g .

$$ R_g = \left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1} $$

End

6. Calculate Anelastic Strain Vector And Update Memory Load Matrix

$$ \bar{\varepsilon}_{s+1} = \left(1-c\right)\frac{E_\infty - E_0}{E_0}\varepsilon_{s+1} - c\sum_{k=1}^{S_t} B_{k+1}\bar{\varepsilon}_{s+1-k} $$

7. Update Time Step And Return To 3rd Step

β_1 / β_2 – Bathe Method

The β_1 / β_2 – Bathe method is an explicit method with two sub-steps. The size of the first substep and second substep are $\left(\gamma\right)\Delta t$ and $\left(1-\gamma\right)\Delta t$, respectively. The displacement, velocity and acceleration relation for first substep at time $\left(s+\gamma\right)\Delta t$ is given as follows:

$$\dot{V}_{s+\gamma} = \dot{V}_s + (\gamma\Delta t)\ddot{V}_s + \beta_1(\gamma\Delta t)\left(\ddot{V}_{s+\gamma} - \ddot{V}_s\right) \tag{29}$$
$$V_{s+\gamma} = V_s + (\gamma\Delta t)\dot{V}_s + 0.5(\gamma\Delta t)^2\ddot{V}_s + \beta_2(\gamma\Delta t)^2\left(\ddot{V}_{s+\gamma} - \ddot{V}_s\right)$$

The displacement, velocity and acceleration relation for second substep at time $(s+1)\Delta t$ is given as follows:

$$\dot{V}_{s+1} = \dot{V}_{s+\gamma} + (1-\gamma)(\Delta t)\ddot{V}_{s+\gamma} + \beta_1(1-\gamma)(\Delta t)\left(\ddot{V}_{s+1} - \ddot{V}_{s+\gamma}\right) \tag{30}$$
$$V_{s+1} = V_{s+\gamma} + (1-\gamma)(\Delta t)\dot{V}_{s+\gamma} + 0.5(1-\gamma)^2(\Delta t)^2\left(\ddot{V}_{s+\gamma}\right)$$
$$+\beta_2(1-\gamma)^2(\Delta t)^2\left(\ddot{V}_{s+1} - \ddot{V}_{s+\gamma}\right)$$

The parameters of the method are β_1, β_2 and γ. In order to solve the equation of motion (Eq.(26)), incremental displacement vectors $\left(\Delta V_{s+\gamma}\right)$ for first substep at time $(s+\gamma)\Delta t$ is given as follows:

$$\Delta V_{s+\gamma} = \left(\left(1/a_1^2\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+\gamma} - M_g\,{}^i\ddot{V}_{s+\gamma} - K_g\,{}^iV_{s+\gamma} - \left(F_{visco}\right)_{s+\gamma}\right\}\ a_1 = (\Delta t/6) \tag{31}$$

Incremental displacement vector $\left(\Delta V_{s+1}\right)$ for second substep at time $(s+1)\Delta t$ is given as follows:

$$\Delta V_{s+1} = \left(\left(1/b_1^2\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1} - M_g\,{}^i\ddot{V}_{s+1} - K_g\,{}^iV_{s+1} - \left(F_{visco}\right)_{s+1}\right\}\ b_1 = (\Delta t/3) \tag{32}$$

Algorithm implementation

The algorithm for the β_1/β_2 – Bathe method is given as follows:

Follow the Step 1 to Step 3 present in the algorithm given for the Newmark method.

6. Calculate The Residual Load Vector R_g With ${}^iV_{s+\gamma} = {}^iV_s$ As Initial Guess and Calculate

${}^i\dot{V}_{s+\gamma}, {}^i\ddot{V}_{s+\gamma}, R_g = \left(F_g\right)_{s+\gamma} - M_g\,{}^i\ddot{V}_{s+\gamma} - K_g\,{}^iV_{s+\gamma} - \left(F_{visco}\right)_{s+\gamma}$

5. While $Norm(R_g) > TOL$ *(Substep-1)*

(a) Calculate the incremental displacement:

$\Delta V_{s+\gamma} = \left(a_1^2 M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+\gamma} - M_g\,{}^i\ddot{V}_{s+\gamma} - K_g\,{}^iV_{s+\gamma} - \left(F_{visco}\right)_{s+\gamma}\right\}$

b. Update the displacement

$V_{s+\gamma} = {}^iV_{s+\gamma} + \Delta V_{s+\gamma}$

c. Update the velocity and acceleration

${}^i\ddot{V}_{s+\gamma} = a_1^2({}^iV_{s+\gamma} - {}^iV_s - a_2\,{}^i\dot{V}_s) - {}^i\ddot{V}_s$

${}^i\dot{V}_{s+\gamma} = {}^i\dot{V}_s + (0.5a_2)({}^i\ddot{V}_{s+\gamma} + {}^i\ddot{V}_s)$

d. Again, calculate the residual R_g.

$R_g = \left(F_g\right)_{s+\gamma} - M_g\,{}^i\ddot{V}_{s+\gamma} - K_g\,{}^iV_{s+\gamma} - \left(F_{visco}\right)_{s+\gamma}$

End

6.Calculate the residual load vector R_g with ${}^iV_{s+1} = {}^iV_{s+\gamma}$ as initial guess and calculate

${}^i\dot{V}_{s+1}, {}^i\ddot{V}_{s+1},$

$R_g = \left(F_g\right)_{s+1} - M_g\,{}^i\ddot{V}_{s+1} - K_g\,{}^iV_{s+1} - \left(F_{visco}\right)_{s+1}$

5. While $Norm(R_g) > TOL$ *(Substep-2)*

(a) Calculate the incremental displacement:

$$\Delta V_{s+1} = \left(\left(1/a_3^{\,2}\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1} - M_g\,{}^i\ddot{V}_{s+1} - K_g\,{}^iV_{s+1} - \left(F_{visco}\right)_{s+1}\right\}$$

(b) Update the displacement

$$V_{s+1} = {}^iV_{s+1} + \Delta V_{s+1}$$

(c) Update the velocity and acceleration

$${}^i\ddot{V}_{s+1} = \left(1/a_3^{\,2}\right)\left({}^iV_{s+1} - {}^iV_s\right) - \left(a_4\right){}^i\dot{V}_s - \left(a_5\right){}^i\dot{V}_{s+\gamma} - \left(a_5\right){}^i\ddot{V}_s - \left(a_6\right){}^i\ddot{V}_{s+\gamma}$$

$${}^i\dot{V}_{s+1} = \left(1/a_3\right)\left({}^iV_{s+1} - {}^iV_s\right) - \left(a_6\right){}^i\dot{V}_s - \left(a_7\right){}^i\dot{V}_{s+\gamma}$$

Where,

$$a_1 = \left(\frac{1}{2\gamma\Delta t}\right), a_2 = \left(\gamma\Delta t\right), a_3 = \left\{\beta_2\left(1-\gamma\right)\Delta t\right\}, a_4 = \left\{\frac{\gamma\left(1-\beta_1\right)+\beta_2\left(1-\gamma\right)}{\left[\beta_2\left(1-\gamma\right)\right]^2\Delta t}\right\}$$

$$a_5 = \left\{\frac{\gamma\left(\beta_1\right)+\left(1-\gamma\right)\left(1-\beta_2\right)}{\left[\beta_2\left(1-\gamma\right)\right]^2\Delta t}\right\}, a_6 = \left\{\frac{\gamma\left(1-\beta_1\right)+\beta_2\left(1-\gamma\right)}{\beta_2\left(1-\gamma\right)}\right\}, a_7 = \left\{\frac{\gamma\left(\beta_1\right)+\left(1-\gamma\right)\left(1-\beta_2\right)}{\beta_2\left(1-\gamma\right)}\right\}$$

(d) Again, calculate the residual R_g .

$$R_g = \left(F_g\right)_{s+1} - M_g\,{}^i\ddot{V}_{s+1} - K_g\,{}^iV_{s+1} - \left(F_{visco}\right)_{s+1}$$

End

7. Calculate anelastic strain vector and update memory load matrix

$$\bar{\varepsilon}_{s+1} = \left(1-c\right)\frac{E_\infty - E_0}{E_0}\varepsilon_{s+1} - c\sum_{k=1}^{S_t} B_{k+1}\bar{\varepsilon}_{s+1-k}$$

8. Update time step and return to 3rd step

TTBDF Method

The TTBDF method is a composite time integration scheme with three step solution methodology. The total time step $\left(\Delta t\right)$ is divided into 3 parts with substep size of $\left(\Delta t/3\right)$.The displacement, velocity and acceleration relation for first substep at time $\left(s+1/3\right)\Delta t$ is given as follows:

$$V_{s+1/3} = V_s + \left(\frac{\Delta t}{6}\right)\left(\dot{V}_s + \dot{V}_{s+1/3}\right)\dot{V}_{s+1/3} = \dot{V}_s + \left(\frac{\Delta t}{6}\right)\left(\ddot{V}_s + \ddot{V}_{s+1/3}\right) \tag{33}$$

The displacement, velocity and acceleration relation for second substep at time $\left(s+2/3\right)\Delta t$ is given as follows:

$$V_{s+2/3} = V_{s+1/3} + \left(\frac{\Delta t}{6}\right)\left(\dot{V}_{s+1/3} + \dot{V}_{s+2/3}\right)\dot{V}_{s+2/3} = \dot{V}_{s+1/3} + \left(\frac{\Delta t}{6}\right)\left(\ddot{V}_{s+1/3} + \ddot{V}_{s+2/3}\right) \tag{34}$$

The displacement, velocity and acceleration relation for third substep at time $\left(s+1\right)\Delta t$ is given as follows:

$$\dot{V}_{s+1} = \left\{\frac{A\left(\theta 1\right)}{\Delta t/3}\right\}V_{s+1} + \left\{\frac{B\left(\theta 1\right)}{\Delta t/3}\right\}V_{s+2/3} + \left\{\frac{C\left(\theta 1\right)}{\Delta t/3}\right\}V_{s+1/3} + \left\{\frac{D\left(\theta 1\right)}{\Delta t/3}\right\}V_s \tag{35}$$

$$\ddot{V}_{s+1/3} = \left\{\frac{A\left(\theta 2\right)}{\Delta t/3}\right\}\dot{V}_{s+1} + \left\{\frac{B\left(\theta 2\right)}{\Delta t/3}\right\}\dot{V}_{s+2/3} + \left\{\frac{C\left(\theta 2\right)}{\Delta t/3}\right\}\dot{V}_{s+1/3} + \left\{\frac{D\left(\theta 2\right)}{\Delta t/3}\right\}\dot{V}_s$$

$$A\left(\theta\right) = \frac{11}{6} - \frac{\theta}{3},\quad B\left(\theta\right) = \theta - 3,\quad C\left(\theta\right) = \frac{3}{2} - \theta,\quad D\left(\theta\right) = \frac{-1}{3} + \frac{\theta}{3}$$

In order to solve the equation of motion (Eq.(26)), incremental displacement vectors $\Delta V_{s+1/3}$ for first substep at time $\left(s+1/3\right)\Delta t$ is given as follows:

$$\Delta V_{s+1/3} = \left(\left(1/a_1^{2}\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1/3} - M_g\,{}^{i}\ddot{V}_{s+1/3} - K_g\,{}^{i}V_{s+1/3} - \left(F_{visco}\right)_{s+1/3}\right\} \quad (36)$$

$$a_1 = \left(\Delta t/6\right)$$

Incremental displacement vector $\Delta V_{s+2/3}$ for second substep at time $\left(s+2/3\right)\Delta t$ is given as follows:

$$\Delta V_{s+2/3} = \left(\left(1/b_1^{2}\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+2/3} - M_g\,{}^{i}\ddot{V}_{s+2/3} - K_g\,{}^{i}V_{s+2/3} - \left(F_{visco}\right)_{s+2/3}\right\}\; b_1 = \left(\Delta t/3\right) \quad (37)$$

Incremental displacement vector ΔV_{s+1} for third substep at time $\left(s+1\right)\Delta t$ is given as follows:

$$\Delta V_{s+1} = \left(a_0 M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}\right\} \quad (38)$$

$$a_0 = A_1\left(\theta 1\right)A_1\left(\theta 2\right)$$

Algorithm implementation

The algorithm for the TTBDF method is given as follows:

Follow the Step 1 to Step3 present in the algorithm given for newmark method.

Calculate the residual load vector R_g with ${}^{i}V_{s+1/3} = {}^{i}V_s$ as initial guess and calculate

${}^{i}\dot{V}_{s+1/3}, {}^{i}\ddot{V}_{s+1/3}$

$$R_g = \left(F_g\right)_{s+1/3} - M_g\,{}^{i}\ddot{V}_{s+1/3} - K_g\,{}^{i}V_{s+1/3} - \left(F_{visco}\right)_{s+1/3}$$

5. While $Norm(R_g) > TOL$ *(Substep-1)*

(a) Calculate the incremental displacement:

$$\Delta V_{s+1/3} = \left(\left(1/a_1^{2}\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1/3} - M_g\,{}^{i}\ddot{V}_{s+1/3} - K_g\,{}^{i}V_{s+1/3} - \left(F_{visco}\right)_{s+1/3}\right\}$$

(b) Update the displacement

$$V_{s+1/3} = {}^{i}V_{s+1/3} + \Delta V_{s+1/3}$$

(c) Update the acceleration and velocity

$${}^{i}\ddot{V}_{s+1/3} = \left(1/a_1^{2}\right)\left({}^{i}V_{s+1/3} - {}^{i}V_s\right) - \left(2/a_1\right){}^{i}V_s - {}^{i}\ddot{V}_s$$

$${}^{i}\dot{V}_{s+1/3} = {}^{i}\dot{V}_s + a_1\left({}^{i}\ddot{V}_s + {}^{i}\ddot{V}_{s+1/3}\right)$$

(d) Again, calculate the residual R_g .

$$R_g = \left(F_g\right)_{s+1/3} - M_g\,{}^{i}\ddot{V}_{s+1/3} - K_g\,{}^{i}V_{s+1/3} - \left(F_{visco}\right)_{s+1/3}$$

End

6.Calculate the residual load vector R_g with ${}^{i}V_{s+2/3} = {}^{i}V_{s+1/3}$ as initial guess and calculate

${}^{i}\dot{V}_{s+2/3}, {}^{i}\ddot{V}_{s+2/3}$

$$R_g = \left(F_g\right)_{s+2/3} - M_g\,{}^{i}\ddot{V}_{s+2/3} - K_g\,{}^{i}V_{s+2/3} - \left(F_v\right)_{s+2/3}$$

7. While $Norm(R_g) > TOL$ *(Substep-2)*

(a) Calculate the incremental displacement:

$$\Delta V_{s+2/3} = \left(\left(1/b_1^2\right)M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+2/3} - M_g\,{}^{i}\ddot{V}_{s+2/3} - K_g\,{}^{i}V_{s+2/3} - \left(F_{visco}\right)_{s+2/3}\right\}$$

(b) Update the displacement

$$V_{s+2/3} = {}^{i}V_{s+2/3} + \Delta V_{s+2/3}$$

(c) Update the acceleration

$${}^{i}\ddot{V}_{s+2/3} = \left(1/a_1^2\right)\left({}^{i}V_{s+2/3} - {}^{i}V_{s+1/3}\right) - \left(2/a_1\right){}^{i}\dot{V}_s - {}^{i}\ddot{V}_s$$

$${}^{i}\dot{V}_{s+2/3} = {}^{i}\dot{V}_{s+1/3} + a_1\left({}^{i}\ddot{V}_{s+1/3} + {}^{i}\ddot{V}_{s+2/3}\right)$$

(d) Again, calculate the residual R_g .

$$R_g = \left(F_g\right)_{s+2/3} - M_g\,{}^{i}\ddot{V}_{s+2/3} - K_g\,{}^{i}V_{s+2/3} - \left(F_{visco}\right)_{s+2/3}$$

End

8.Calculate the residual load vector R_g with ${}^{i}V_{s+1} = {}^{i}V_{s+2/3}$ as initial guess and calculate

${}^{i}\dot{V}_{s+1}, {}^{i}\ddot{V}_{s+1}$

$$R_g = \left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}$$

9. While $Norm(R_g) > TOL$ *(Substep-3)*

(a) Calculate the incremental displacement:

$$\Delta V_{s+1} = \left(a_0 M_g + K_{gT}\right)^{-1}\left\{\left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}\right\}$$

(b) Update the displacement

$$V_{s+1} = {}^{i}V_{s+1} + \Delta V_{s+1}$$

(c) Update the acceleration and velocity

$${}^{i}\ddot{V}_{s+1/3} = A_1\,{}^{i}\dot{V}_{s+1} + A_2\,{}^{i}\dot{V}_{s+2/3} + A_3\,{}^{i}\dot{V}_{s+1/3} + A_4\,{}^{i}\dot{V}_s$$

$${}^{i}\dot{V}_{s+1/3} = A_1\,{}^{i}V_{s+1} + A_2\,{}^{i}V_{s+2/3} + A_3\,{}^{i}V_{s+1/3} + A_4\,{}^{i}V_s$$

(d) Again, calculate the residual R_g .

$R_g = \left(F_g\right)_{s+1} - M_g\,{}^{i}\ddot{V}_{s+1} - K_g\,{}^{i}V_{s+1} - \left(F_{visco}\right)_{s+1}$ End

10.Calculate anelastic strain vector and update memory load matrix

$$\bar{\varepsilon}_{s+1} = \left(1-c\right)\frac{E_\infty - E_0}{E_0}\varepsilon_{s+1} - c\sum_{k=1}^{S_t} B_{k+1}\bar{\varepsilon}_{s+1-k}$$

11. Update time step and return to 3rd step

RESULTS AND DISCUSSION

The finite element (FE) formulation is initially validated through comparison with a reference problem outlined in the study by Galucio et al [13]. The verification problem involves a cantilever sandwich beam configuration with a viscoelastic core. In order to verify the FE formulation, the value of the external mass $\left(m_l\right)$ is considered as zero. The curved panel is analogously treated as a beam, with the radius approaching infinity. This beam is then modeled as a cantilever, with designated parameters: top and bottom layer thickness denoted as h_1 and core thickness denoted as h_2 . The beam's top and bottom layers consist of aluminum, while the core comprises the viscoelastic material ISD112. Mechanical properties for bottom and top face is $\rho = 2690 Kg/m^3$, $\nu = 0.345$ and $E = 70.3\times10^3 MPa$. Also, the mechanical properties of the visco-elastic core is $\rho = 1600 Kg/m^3$, $\nu = 0.5$, $E_0 = 1.5 Mpa$, $\alpha = 0.7915$ $E_\infty = 69.9495 MPa$, $\tau = 1.4052\times10^{-2} ms$ .The geometric properties of the beam are length ($b = 10mm$) ,thickness of the top and bottom layer of beam ($h_1 = 1mm$), thickness of the core is

$h_2 = 0.2mm$. Subsequently, a triangular impulse load is applied at the beam's free end, and the resulting displacement response at the tip is compared with that of the literature problem, as depicted in Fig. 3. The agreement between the response obtained through the nonlinear formulation and that of the reference problem confirms the accuracy of the FE formulation. Thus, it can be inferred that the FE model has been appropriately configured and validated. Further, the verified FE model is used to evaluate the dynamic response of the curved panel under external force for present three-time integration schemes. The external force is applied as suddenly applied force with force parameters values as $F^i = m_l g$, $F^c = 5N$ and $\omega = (2\pi f)rad/\sec$ where, $m_l = 20Kg,\ g = 9.81m/s^2,\ f = 10hz$ .The panel's dimensions are $R_1 = 858.1\ mm,\ R_2 = 859.3mm,\ R_3 = 861.7mm,\ R_4 = 862.9mm,\ b = 50mm,$ $h_1 = 1.2mm,\ h_2 = 2.4mm$. Next, the metal and viscoelastic material is considered as aluminium and nitrile rubber. The properties of the aluminium is $\rho = 2310Kg/m^3, \nu = 0.31$ and $E = 70GPa$. Also, the mechanical properties of the nitrile rubber is $\rho = 1163Kg/m^3, \nu = 0.49,\ E_0 = 9.2Mpa,\ \alpha = 0.1924$ $E_\infty = 104.2MPa,\ \tau = 5.198\times10^{-6}s$.

Next, the time integration parameters of all three-time integration schemes are set. The time integration parameters for newmark method are $\delta = 0.5$ and $\alpha = 0.25$, while, the time integration parameters for $\beta_1/\beta_2 -$ bathe method are $\beta_1 = 0.33, \beta_2 = 1-\beta_1$ and $\gamma = 0.5$. Also, the time integration parameters for TTBDF method are $\theta 1 = 0.75$ and $\theta 2 = 0.75$.Further, the total time step (Δt) for newmark method , $\beta_1/\beta_2 -$bathe method and TTBDF method are set as $1\times10^{-3}s,\ 2\times10^{-3}s$ and $3\times10^{-3}s$,respectively. Total time step for each method is taken in such a way that each substep has equal time span. The displacement response of the top middle of the panel is evaluated for 3 s. The displacement-time response for newmark method, $\beta_1/\beta_2 -$bathe method and TTBDF method are shown in Figure 4(a)–(c) , respectively. From the comparison of the responses as shown in Figure 4 (d), it can be inferred that the newmark method, $\beta_1/\beta_2 -$bathe method and TTBDF method accurately captures the forcing frequency of the external force excitation

But the displacements of the panel slightly differ in magnitude for all the three methods for the given problem with specific boundary and loading conditions. The difference in the displacements of the panel is very small. Numerical damping for all three methods is kept at zero by setting the respective time integration parameters accordingly. The slight variation in the responses of the panel may be due to the application of the Grunwald definition, (Schmidt and Gaul) [4]. The anelastic strains are updated at the end of the complete time step for $\beta_1/\beta_2 -$ bathe and TTBDF method. This may results in slight change in the damping and can be seen in the Figure 4 (d). Since, the variation in the displacement responses is insignificant, it can be said that the $\beta_1/\beta_2 -$ bathe and TTBDF method are coupled with FE model correctly.

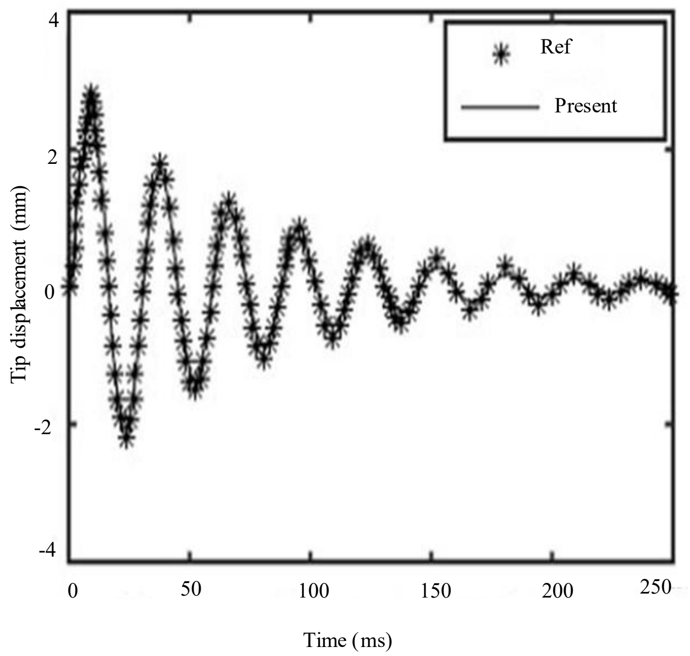
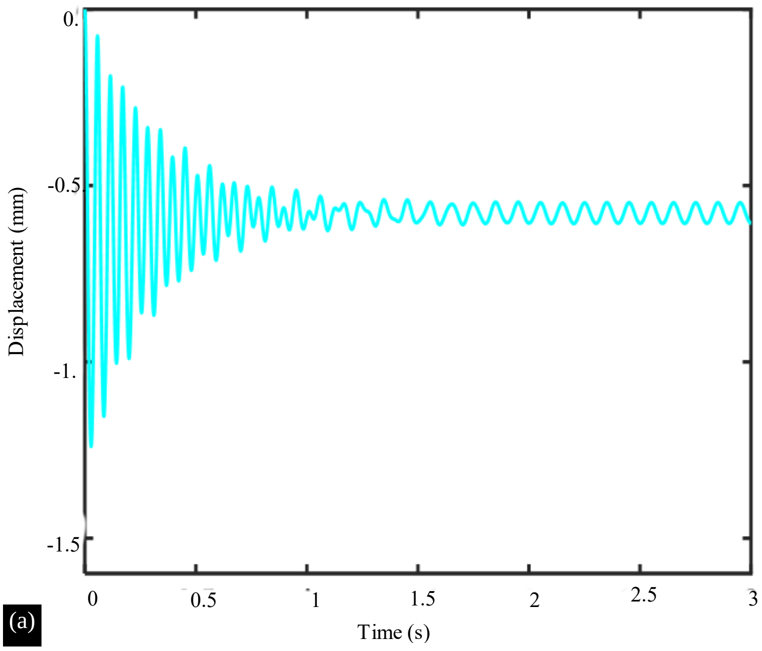


Figure 3. Validation of the current FE model based on FOD constitutive relation using Newmark method for the sandwich beam (Ref. Galucio et al.,[12].



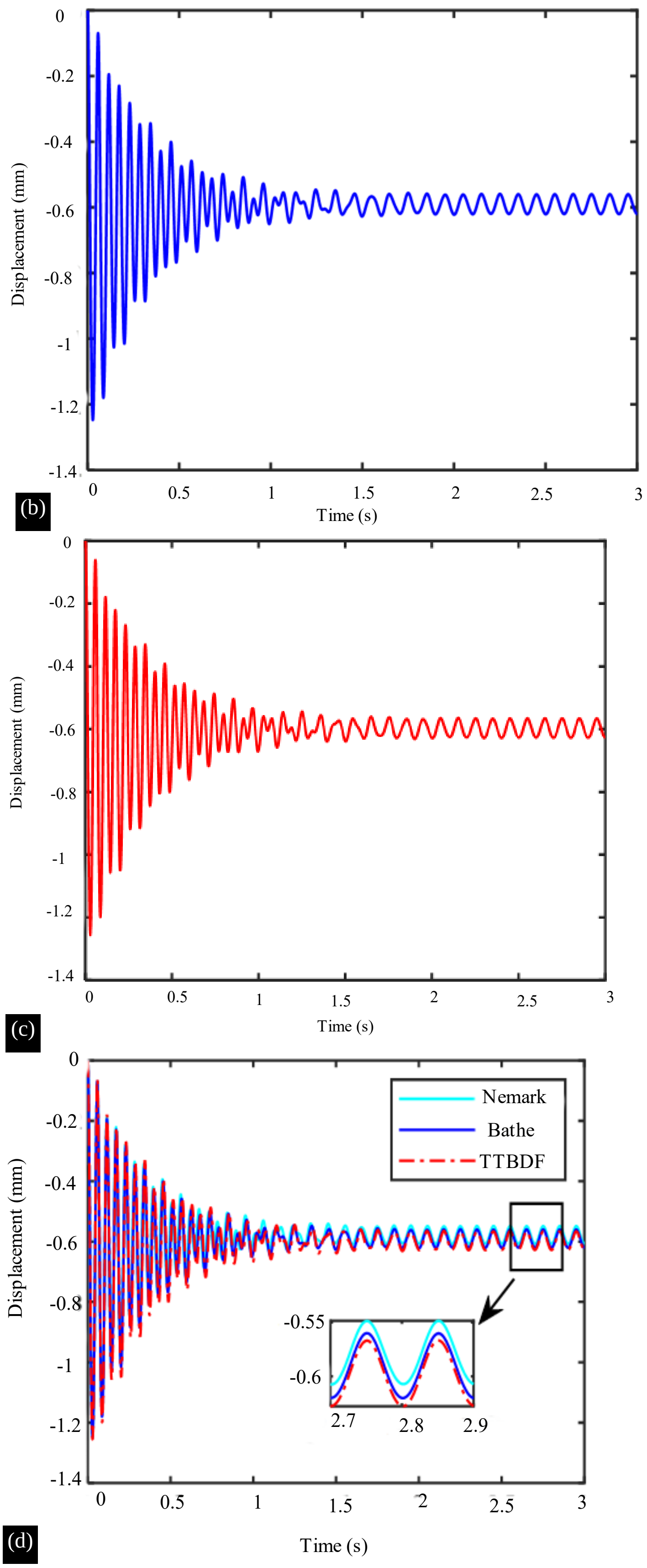


Figure 4. Displacement-time response for (a) Newmark method, (b) Bathe method (c) TTBDF method and (d) comparison of responses.

CONCLUSION

In this study, a composite curved panel with a nitrile rubber core constrained by aluminium panels was analyzed. The nitrile rubber was characterized by a four-parameter fractional order derivative model, and its dynamic mechanical properties were experimentally determined through dynamic mechanical analysis. A non-linear dynamic finite element model was derived for the four-parameter fractional order derivative viscoelastic model using a Newton-Raphson iterative method. Then, various time integration schemes, such as the Newmark implicit method, β_1/β_2-Bathe explicit method, and TTBDF composite method, were incorporated into the finite element model. The algorithms for implementing these time integration schemes were explained in detail. The numerical solution obtained from the non-linear finite element model was verified, taking into account the influence of anelastic strain history in the tangent stiffness matrix. The assessment of the nonlinear response of the composite curved panel involved analyzing it through the Newmark, β_1/β_2-Bathe, and TTBDF methods, followed by a comparative analysis of the outcomes. It was observed that all three methods effectively captured the forcing frequency, indicating their reliability in that aspect. However, when examining the displacement-time response, minor inconsistencies were noted among the methods. These variations, although negligible, could possibly be attributed to the update of the anelastic strains at the end of the complete time step.

APPENDIX

$$^{i}\varepsilon_l=\left[\frac{\partial^i u}{\partial x}\frac{\partial^i w}{\partial z}\left(\frac{\partial^i u}{\partial z}+\frac{\partial^i w}{\partial x}\right)\right]^T \qquad ^{i}\varepsilon_{n1}=\left[\frac{1}{2}\left(\frac{\partial^i u}{\partial x}\right)^2\frac{1}{2}\left(\frac{\partial^i u}{\partial z}\right)^2\left(\frac{\partial^i u}{\partial x}\frac{\partial^i u}{\partial z}\right)\right]^T$$

$$^{i}\varepsilon_{n2}=\left[\frac{1}{2}\left(\frac{\partial^i w}{\partial x}\right)^2\frac{1}{2}\left(\frac{\partial^i w}{\partial z}\right)^2\left(\frac{\partial^i w}{\partial x}\frac{\partial^i w}{\partial z}\right)\right]^T \qquad \Delta\varepsilon_l=\left[\frac{\partial\Delta u}{\partial x}\frac{\partial\Delta w}{\partial z}\left(\frac{\partial\Delta u}{\partial z}+\frac{\partial\Delta w}{\partial x}\right)\right]^T$$

$$\Delta\varepsilon_{n1}=\left[\frac{1}{2}\left(\frac{\partial\Delta u}{\partial x}\right)^2\frac{1}{2}\left(\frac{\partial\Delta u}{\partial z}\right)^2\left(\frac{\partial\Delta u}{\partial x}\frac{\partial\Delta u}{\partial z}\right)\right]^T \qquad \Delta\varepsilon_{n2}=\left[\frac{1}{2}\left(\frac{\partial\Delta w}{\partial x}\right)^2\frac{1}{2}\left(\frac{\partial\Delta w}{\partial z}\right)^2\left(\frac{\partial\Delta w}{\partial x}\frac{\partial\Delta w}{\partial z}\right)\right]^T$$

$$\Delta\varepsilon_{nn1}=\left[\left(\frac{\partial^i u}{\partial x}\frac{\partial\Delta u}{\partial x}\right)\left(\frac{\partial^i u}{\partial z}\frac{\partial\Delta u}{\partial z}\right)\left(\frac{\partial^i u}{\partial x}\frac{\partial\Delta u}{\partial z}+\frac{\partial\Delta u}{\partial x}\frac{\partial^i u}{\partial z}\right)\right]^T$$

$$\Delta\varepsilon_{nn2}=\left[\left(\frac{\partial^i w}{\partial x}\frac{\partial\Delta w}{\partial x}\right)\left(\frac{\partial^i w}{\partial z}\frac{\partial\Delta w}{\partial z}\right)\left(\frac{\partial^i w}{\partial x}\frac{\partial\Delta w}{\partial z}+\frac{\partial\Delta w}{\partial x}\frac{^i w}{\partial z}\right)\right]^T \quad B_{ux}=\left[\frac{\partial N_i}{\partial x}\quad 0\right] \qquad B_{uz}=\left[\frac{\partial N_i}{\partial z}\quad 0\right]$$

$$B_{wx}=\left[0\quad \frac{\partial N_i}{\partial x}\right] \qquad B_{wz}=\left[0\quad \frac{\partial N_i}{\partial z}\right]$$

$$B_l=\begin{bmatrix}\frac{\partial N_i}{\partial x} & 0\\ 0 & \frac{\partial N_i}{\partial z}\\ \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial x}\end{bmatrix} \qquad B_{n1}=\begin{bmatrix}\tfrac{1}{2}B_{ux}B_{uxI}\\ \tfrac{1}{2}B_{uz}B_{uzI}\\ B_{ux}B_{uzI}\end{bmatrix} \qquad B_{n2}=\begin{bmatrix}\tfrac{1}{2}B_{wx}B_{wxI}\\ \tfrac{1}{2}B_{wz}B_{wzI}\\ B_{wx}B_{wzI}\end{bmatrix} \quad B_{nn1}=\begin{bmatrix}B_{ux}B_{uxI}\\ B_{uz}B_{uzI}\\ B_{uxz}\end{bmatrix}$$

$$B_{nn2}=\begin{bmatrix}B_{wx}B_{wxI}\\ B_{wz}B_{wzI}\\ B_{wxz}\end{bmatrix} \quad B_g=\begin{bmatrix}\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial z} & 0 & 0\\ 0 & 0 & \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial z}\end{bmatrix}^T \qquad B_{uxI}=I\otimes B_{ux} \qquad B_{uzI}=I\otimes B_{uz}$$

$$B_{wxI} = I \otimes B_{wx} \quad B_{wzI} = I \otimes B_{wz} \quad S_m = \begin{bmatrix} \sigma_x & \tau_{xz} & 0 & 0 \\ \tau_{xz} & \sigma_z & 0 & 0 \\ 0 & 0 & \sigma_x & \tau_{xz} \\ 0 & 0 & \tau_{xz} & \sigma_z \end{bmatrix} \quad {}^iV_{eP} = I \otimes {}^iV_e$$

NOMENCLATURE

σ	Stress vector (N/m^2)
ε	Strain vector (m/m)
$\bar{\varepsilon}$	Anelastic strain vector (m/m)
C	Constitutive matrix
E_0	Relaxed modulus (N/m^2)
E_∞	Instantaneous modulus (N/m^2)
α	**Fractional order**
τ	Relaxation time (s)
ρ	Density (Kg/m^3)
v	Poisson's ratio
u	Displacement in x-direction (m)
w	Displacement in y-direction (m)
F^i	Static force (N)
F^c	Dynamic force (N)
ω	Angular frequency $(rad/\sec)$
R_i	Radius (m)
θ	Angle of the panel
b	Width of the panel (m)
h_1	Thickness of the metal panel (m)
h_2	Thickness of the viscoelastic panel (m)
$E^*(\omega)$	Complex modulus (N/m^2)
E'	Storage modulus (N/m^2)
E''	Loss modulus (N/m^2)
m_l	External mass (Kg)
${}^iV$	Global displacement vector at reference state (m)
${}^i\dot{V}$	Global velocity vector at reference state (m/s)
${}^i\ddot{V}$	Global acceleration vector at reference state (m/s^2)
ΔV	Incremental displacement vector at incremental state (m)
$\Delta\ddot{V}$	Incremental acceleration vector at incremental state (m/s^2)
R_g	Residual vector
M_g	Global mass matrices
K_g	Global stiffness matrices
K_{gT}	Global tangent stiffness matrices
F_{visco}	Global viscoelastic load vector
F_g	Global force vector

Δt	Step size (s)
B_{k+1}	Grunwald constant
S_t	Number of time divisions
δ, α_n	Newmark time integration parameters
β_1, β_2, γ	β_1 / β_2 – Bathe time integration parameters
$\theta 1, \theta 2$	TTBDF time integration parameters

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