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Comparative Study of G-metric Space, Partial Metric Space, and Intuitionistic Fuzzy Metric Space Spaces

Ranganath M.[1,*]

Abstract

In advanced mathematics, metric spaces are fundamental for measuring the distance between points within a set, facilitating the analysis of complex entities like sequences, sets, and functions beyond basic objects such as integers and vectors. A metric space consists of a non-empty set (X) paired with a metric, a two-variable function that measures the distance between elements. To address different mathematical structures and needs, various types of metric spaces have been developed, including G-metric spaces, partial metric spaces, and intuitionistic fuzzy metric spaces, each offering distinct characteristics and applications. G-metric spaces generalizes the classical metric space by utilizing a three-variable function that adheres to specific axioms, making it suitable for exploring structures that require a broader notion of distance. Partial metric spaces, in contrast, permit self-distance, where the distance between an element and itself may not be zero. This property makes them particularly valuable in fields like computer science and domain theory, which often deal with self-referential or recursive constructs. Intuitionistic Fuzzy Metric Spaces add further complexity by integrating fuzzy logic, which addresses uncertainty and imprecision. These spaces employ both membership and non-membership functions to define distances, offering a more adaptable approach to evaluating relationships between elements. This paper offers a detailed comparison of these metric spaces, exploring their foundational properties, axioms, and practical applications. It examines the contexts in which each type of metric space is most effective, emphasizing their respective strengths and limitations. By contrasting these spaces, the paper clarifies their interrelationships and provides comparative results that highlight their distinctions and commonalities. This analysis not only deepens the theoretical understanding of metric spaces but also aids in selecting the most suitable metric space for mathematical or applied scenarios, thereby contributing significantly to the field of metric space theory.

Keywords: Intuitionistic metric space, fuzzy metric space, G-metric space, partial metric space

INTRODUCTION

In mathematics, especially in the areas of topology and analysis, the metric space is a basic idea. It is a set that has a built-in function that establishes the separation between any two set items. A controlled metric space is a specialized concept in mathematics that is used to deal with issues related to the "control" of distances and the behavior of sequences or functions within a metric space. This concept is often used in the studies of coarse geometry, large-scale geometry, and geometric group theory.

***Author for Correspondence**
Ranganath M.
E-mail: rmrgranganath@gmail.com

[1]Principal, R Muddurangegowda College of Education, Gurukula Badavane, Jyothinagara, Sira Town, Karnataka, India

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REVIEW

Zadeh created the concept of fuzzy sets in 1965 to depict the ambiguity of daily life. After Zadeh's initial but very important work, numerous efforts have been made to obtain fuzzy analogs for classical theories. Significant progress has been made in the field of fuzzy topology; significant progress has been done [1]. Acquiring a suitable notion of the fuzzy metric space is among the most relevant issues in the fuzzy topology. Numerous writers have addressed this issue from various perspectives. The fuzzy metric space is a concept that George and Veeramani presented and explored.

Kramosil and Michalek studied fuzzy metric space, which is a subclass of probabilistic metric spaces of a close relation. George and Veeramani adapted the fuzzy metric space concept, first proposed by Kramosil and Michalek. In the modified fuzzy metric space, they obtained a Hausdorff and first-countable topology. This type of fuzzy metric space is highly significant in quantum particle physics, especially for string and ∞ theories. Since then, Gregori and Romaguera have demonstrated that in the meaning of George and Veeramani, the topology induced by a fuzzy metric space is metrizable.

In 1951, Merger established the concept of a statistical metric. Kramosil and Michalek proposed the concept of a fuzzy metric based on the statistical metric concept. This measure is referred to as the KM-fuzzy measure. All that separates a fuzzy metric from a statistical metric is how it is defined and interpreted.

George and Veeramani modified the fuzzy metric in 1994. known as the KM-fuzzy metric. Many real-world instances of fuzzy measurements have been provided by this modification. Studying induced topological structures appears to be better suited to fuzzy metrics. In addition to piquing the interest of numerous scholars studying the theoretical aspects of fuzzy metrics, mathematicians working in physics, mathematics, and applied mathematics have also been greatly intrigued by the topological and sequential characteristics of fuzzy metric spaces, completeness, and fixed points of mappings [2].

Definition

In the fields of topology and analysis, metric space is an essential notion in mathematics. The idea of the distance between points in a set can be discussed and examined using its framework.

Metric space is a pair (X, d), where

- X is a non-empty set
- d is a function d: $X \times X \to R$ (called a metric or distance function) that satisfies the following properties for all x, y, z $\in$ X.
  1. Non-negativity: $d(x, y) \geq 0$ $d(x, y)=0$ if and only if $x=y$.
  2. Symmetry: $d(x, y)=d(y, x)$.
  3. Triangle Inequality: $d(x, z) \leq d(x, y)+d(y, z)$

Fuzzy Set

A fuzzy set is described as a collection of pairs that each include a specific element of the universe X and its degree of membership, denoted by a value of [0.1] that represents the extent to which property x possesses $A= \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), (x_3, \mu_A(x_3)) (x_n, \mu_A(x_n))\}$.

Fixed Point

In mathematics, a value that remains unchanged after a specific transformation is referred to as a fixed point (invariant point). A fixed point is an element that a function maps to itself, specifically for mathematical functions [3].

Metric Space

A non-empty set and associated metric comprise a metric space. It is common to denote "metric space" with (X, p). Property (iv) defines the triangle inequality for the metric. The set R of all real numbers with $p(x, y) = | x - y |$ is a classic example of metric space. The set τ is known as the topology on X, and the space (X, τ) is known as the topological space. The open sets are the constituents of τ. A metric space is a set X and a function d: $X \times X \to R+ \cup \{0\}$ called the "metric" which takes in two elements from the set and pops out a non-negative real number [4].

Definition

A pair (X, d) that satisfies the following axioms is a metric space: X is a set and d is a real-valued mapping on X × X.

$d(x, y) \geq 0$ for all $x, y \in X$

$d(x, y) = 0$ if and only if $x = y$

$d(x, y) = d(y, x)$ for all $x, y \in X$

$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Here, d is the metric on X; that is, $d(x, y)$ is the distance from x to y.

This Axiom is known as the triangular inequality axiom. It is customary to refer to X as the metric space, even if, in pure mathematical terms, the pair (X, d) is the metric space if metric d is understood from the context [5].

In this scenario, we say that Y inherits the metric of X, as it is obvious that if (X, d) is a metric space, then (Y, d) is likewise [6].

Below are some metric spaces and their comparative properties

G-Metric Space

Given a set $X \neq \emptyset$, a function $G{:}X \times X \times X \rightarrow \mathbb{R}$ is called metric-G on X if for every $x, y, z \in X$ satisfy these following conditions:

i. $(x, y, z) = 0$ $x = y = z$.
ii. $G(x, x, y) > 0$ $x \neq y$
iii. $G(x, y, z) = G(y, x, z) = G(x, z, y)$
iv. $(x, x, y) \leq (x, y, z)$ $x, y, z \in X$ $y \neq z$.
v. $(x, y, z) \leq (x, a, a) + (a, y, z)$ $x, y, z.$ $a \in X$. Hence Set X which follows conditions of G-metric on X is called G-metric space, and it is denoted by (X, G).
vi. Let (X, G) be a G-metric space, then for any x, y, z, and a in X it follows that:
    a. $(x, y, z) \leq (x, x, y) + (x, x, z)$
    b. $(x, y, y) \leq 2(y, x, x)$
vii. For any metric space (X, d), the following function is a G-metric on X Gd (x_1, x_2, x_3) $= \{(x_1, x_2), d(x_2, x_3), d(x_3, x_1)\}$.

Partial Metric Space

A partial metric (PM) denoted by p on a non-empty set is a function p: X × X → R +(set of non-negative reals), such that for any x, y, z ∈ X, [7]

i. $p(x, x) \leq p(x, y)$.
ii. If $p(x, x) = p(x, y) = p(y, y)$, $x = y$.
iii. $p(x, y) = p(y, x)$.
iv. iv. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$. Here, (X, p) is called a partial metric space, and is denoted by PMS.

Intuitionistic Fuzzy Metric Space Definition

A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following condition:

i. $*$ is associative and commutative;
ii. $*$ is continuous;
iii. is identity $a * 1 = a$, for all $a \in [0, 1]$;
iv. $a * b \leq c * d$, where $a \leq c$ and $b \leq d$, for each a, b, c, and $d \in [0, 1]$.

Comparative Study

A comparative study among G-metric spaces, partial metric spaces, and intuitionistic fuzzy metric spaces involves analyzing their definitions, properties, and applications. These spaces are essential in various areas of mathematics and their applications, particularly in fixed-point theory, analysis, and fuzzy logic (Table 1) [8].

G-Metric Space

Definition: A G-metric space is a generalization of a metric space defined using a function G, which takes three points instead of two. Specifically, a G-metric on a set X is a function G: X→[0,∞] that satisfies the following properties for all x, y, z, u ∈ X [9].

1. G(x, y, z)= 0 if and only if x=y=z.
2. G(x, x, y)≤G(x, y, y)
3. G(x, y, z)=G(x, z, y) (symmetry in all arguments).
4. G(x, y, z)≤G(x, y, u)+G(x, u, z)+G(u, y, z) (triangular inequality).

Properties

- Generalizes metric spaces by considering triplets of points.
- Used in fixed-point theory and the study of multi-valued mappings.

Applications

- Analysis of algorithms and data structures.
- Studying complex dynamical systems.
- Fixed-point theory.

Partial Metric Space

Definition: A partial metric space is a generalization of a metric space where the self-distance can be non-zero [10, 11]. A partial metric on a set X is a function p: X×X→[0,∞] that satisfies the following properties for all x, y, z ∈ X:

1. p(x, x)≤p(x, y) (small self-distance).
2. p(x, x)=p(y, y)=p(x, y) if and only if x=y.
3. p(x, y)=p(y, x) (symmetry).
4. p(x, y)≤p(x, z)+p(z, y)−p(z, z) (triangle inequality).

Table 1. Space is a generalization of a metric space defined using a function.

Aspect	G-metric space	Partial metric space	Intuitionistic fuzzy metric space
Definition	Generalizes metric spaces to three points	Generalizes metric spaces with non-zero self-distance	Combines fuzzy logic with metric space concepts
Key Properties	Symmetry, triangle inequality with triplets	Non-zero self-distance, modified triangle inequality	Membership and non-membership degrees, fuzzy logic
Complexity	More complex due to triplet relationships	Moderate complexity, simpler than G-metric	Higher complexity due to fuzziness
Accuracy	High for certain applications	High in specific computational models	High in uncertain and imprecise environments
Ease of Use	Moderate, requires an understanding of triplet relationships	Relatively easy, and intuitive for computer science applications	Moderate to difficult due to fuzzy components
Applications	Fixed-point theory, complex systems	Domain theory, concurrent computations	Decision-making, AI, pattern recognition
Flexibility	Flexible in handling complex interactions	Flexible in computer science applications	Highly flexible in uncertain environments

Properties

- It allows self-distance, providing a measure of similarity or closeness, even for identical elements.
- Extending the concept of a metric space to deal with situations in which traditional metrics fail.

Applications

- Domain theory in computer science.
- Modeling of concurrent computations.
- Analysis of algorithms and program semantics.

Intuitionistic Fuzzy Metric Space

Definition: An intuitionistic fuzzy metric space combines the concepts of fuzzy sets and metric spaces, incorporating degrees of membership and non-membership [12, 13]. An intuitionistic fuzzy metric on a set X is a pair of functions $(M, N):(X\times 0,\infty)\to[0, 1]$ such that for all x, y, z, ∈X and t, s>0:

1. $M(x, y, t)+N(x, y, t)\leq 1$ for all $t>0$.
2. $M(x, y, t)>0$ and $N(x, y, t)<1$.
3. $M(x, y, t)=1$ if and only if $x=y$.
4. $M(x, y, t)=M(y, x, t)$ and $N(x, y, t)=N(y, x, t)$.
5. $M(x, y, t)$ is non-decreasing in t and $\lim_{t\to\infty} M(x, y, t)=1$.
6. $M(x, y, t+s)\geq M(x, z, t)$ and $N(x, y, t+s)\leq N(x, z, t)$.

Properties

- Combines the properties of metric spaces with fuzzy logic.
- Handles uncertainty and imprecision effectively.
- Provides a more flexible framework compared to classical metric spaces.

Applications

- Decision-making under uncertainty.
- Pattern recognition and image processing.
- Theoretical computer science and artificial intelligence.

Comparative Analysis

G-metric space, Partial metric space and Intuitionistic fuzzy metric space.

CONCLUSION

Each type of metric space has unique properties and is suitable for different applications. G-metric spaces are useful for complex systems and fixed-point theory, partial metric spaces are valuable in computational models and domain theory, and intuitionistic fuzzy metric spaces are ideal for dealing with uncertainty and imprecision in various scientific fields. The choice of metric space to use depends on the specific requirements and nature of the problem at hand. The choice of a specific type of metric space depends on the needs of the problem at hand. Each metric space provides a distinct set of perspectives and tools that can be adapted to address the specific nature of a task, whether it involves managing complex relationships, dealing with recursive structures, or handling uncertain data. By understanding the unique properties and benefits of each metric space, researchers and practitioners can select the most suitable framework, thereby improving the effectiveness and precision of mathematical modeling and analysis in various scientific and engineering fields.

REFERENCES

1. Zadeh LA. Fuzzy sets. Information and Control. 1965;8:338–53. DOI: 10.1016/S0019-9958(65)90241-X.
2. Rudin W. Principles of Mathematical Analysis. New Delhi: McGraw Hill Education; 2013.
3. Bridson MR, Haefliger A. Metric Spaces of Non-Positive Curvature. Berlin, Heidelberg: Springer-Verlag; 1999. DOI: https://doi.org/10.1007/978-3-662-12494-9.

4. Wainger S. A metric space approach to some classical problems in harmonic analysis. Trans Am Math Soc. 1966;124:241–54.
5. Rockafellar RT. On the maximality of sums of nonlinear monotone operators. Trans Am Math Soc. 1970;149:75–88. DOI: 10.1090/S0002-9947-1970-0282272-5.
6. Roe J. The role of metric spaces in analysis and geometry. Bull Am Math Soc. 1995;32:146–55.
7. Kapovitch V. Gromov-Hausdorff convergence of metric spaces. Geom Funct Anal. 2017;27: 685–738.
8. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning—I. Information Sci. 1975;8:199–249. DOI: 10.1016/0020-0255(75)90036-5.
9. Dubois D, Prade H. Fuzzy Sets and Systems: Theory and Applications. Mathematics in Science and Engineering. Vol. 144. New York: Academic Press, Inc.; 1980.
10. Zimmermann HJ. A review of fuzzy set theory and its applications. Decis Support Syst. 1992;8:191–204.
11. Ruspini EH. Fuzzy logic and approximate reasoning. Information Sci. 1969;2:211–53.
12. Pal SK, King RA. On edge detection of X-ray images using fuzzy sets. IEEE Trans Pattern Anal Mach Intell. 1983 Jan;PAMI-5(1):69–77. DOI: 10.1109/TPAMI.1983.4767347.
13. Buckley JJ. The fuzzy mathematics of finance. Fuzzy Sets Syst. 1987;21:257–73. DOI: 10.1016/0165-0114(87)90128-X.

