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Exploration of Partial Order Structures in Menger Spaces Properties Characterizations and Applications

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Abstract

Partial order structures play a crucial role in understanding the intricate relationships within mathematical spaces. In this paper, we delve into the realm of Menger spaces and investigate their properties through the lens of partial orders. Menger spaces, a generalization of metric spaces, possess unique characteristics that can be further elucidated by considering partial order structures. Through rigorous analysis, we explore various properties of partial order Menger spaces, including topological properties, convergence concepts, and structural characterizations. Additionally, we investigate the applications of partial order Menger spaces in diverse mathematical contexts, such as optimization problems, graph theory, and mathematical modeling. Our study not only contributes to the theoretical understanding of Menger spaces but also sheds light on the practical implications of incorporating partial order structures in mathematical analysis and problem-solving.

Keywords: Partial order, Menger spaces, Topological properties, Convergence

INTRODUCTION

The Banach contraction mapping principle is a cornerstone in analysis, with profound implications across various fields. This principle has been widely extended and adapted, particularly in the study of fixed-point theory. One significant extension is the work of Bhaskar and Lakshmikantam, who explored mixed monotone mappings in partially ordered metric spaces, laying a foundation for further developments in this area [1]. Building on this foundation, our research seeks to generalize and enrich these results, delving into the complexities of contraction mappings and their broad applications.

Menger's concept of probabilistic metric spaces [2], expanded upon by Schweizer and Sklar [3], and others [4], has also played a crucial role in extending fixed-point theory into more generalized and probabilistic contexts. This approach has been further explored by various scholars who investigated the behavior of fixed points in Menger spaces [5], including the study of compatible mappings [6] and common fixed points [7]. These investigations underscore the applicability of contraction principles to Menger spaces and their significance in the broader framework of nonlinear analysis.

Works by Burgić, Kalabušić, and Kulenović [8], as well as by Luong and Thuan [9], have advanced the understanding of mixed monotone mappings and their global attractivity in partially ordered metric spaces. These studies, along with others focused on generalized contractions [10, 11], contribute to a growing body of knowledge that highlights the intricate relationship between contraction mappings and fixed-point theory in both deterministic and probabilistic settings.

Our work aims to further these developments by exploring the intricacies of contraction mappings in

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partially ordered metric spaces, with an emphasis on the existence and uniqueness of fixed points. By building on the contributions of previous research [12, 11], we hope to unveil new insights that enhance both the theoretical and practical understanding of these mappings, reinforcing their relevance across various mathematical contexts.

Theorem 1.1 [25]

Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $T: X \times X \rightarrow X$ be a mapping having the mixed monotone property on X. Assume that there exists a,

$k\epsilon[0,1)$ Such that $d(T(p,q), T(r,s) \leq [k\ d(p,r) + d(q,s)/2\ , \forall\ r \leq p\ , s \leq q$

Suppose

i. T is continuous.
ii. $\{a_n\}$, $\{b_n\}$ are sequences in X then $a_n \longrightarrow a\ , b_n \longrightarrow b$ are a cauchy's sequence.

Then there exist $a_0\ , b_0 \epsilon X$ s.t

$a_0 \leq T(a_0\ , b_0)$ and $a_0 \leq T(b_0\ , a_0) \leq b_0$,

then there exist $a, b\epsilon X$ such that $T(a,b) = a$,and $T(b,a) = b$.

Proof

We generalized the result of Bhaskar and Lakshmikantam

Let $(X, \leq)$ be a partially ordered set and T & g are two Mapping of of such as $T: X \times X \rightarrow X$ and $g: X \times X \rightarrow X$ is called to commute with g if $T(g(a), g(b)) = g(T(a,b)$

for all $a, b\epsilon X$ .T is said to be have mixed g −monotone property non-decreasing and non-increasing of T for any $a, b\epsilon X$

a_1,$a_2 \epsilon X$, therefore $g(a_1) \leq g(a_2) \Longrightarrow T(a_1$,b)$\leq T(a_2, b)$

Similar case for other term

b_1,$b_2 \epsilon X$, therefore $g(b_1) \leq g(b_2) \Longrightarrow T(b_1$,a)$\leq T(b_2, a)$

A element $(a, b)\epsilon X \times X$ is known as coupled coincidence point of the Mapp $T\ and\ g$ if

$g(a) = T(a,b)$ and $g(b) = T(b,a)$

Lakshmikantam and Ciric proved the theorem that extended and improved Theorem 1.1 of Bhaskar and Lakshmikantam.

Lemma 1.1

Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space if $T: X \times X \rightarrow X$ is a mixed monotone mapping and there exist sequences $\{a_n\}$, $\{b_n\}$ in X such that $a_n \longrightarrow a\ , b_n \longrightarrow b$, then the sequences $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences.

Proof

Given that $\{a_n\}$ and $\{b_n\}$ converges to a and b , respectively, we need to show that both sequences are Cauchy.

Since d is a metric, we know that for every $\epsilon > 0$,there exist $N \in$ N such that for all $m, n \geq N$

We have $d(a_n, a_m\ ) < \epsilon$ and $d(b_n, b_m\ ) < \epsilon$.

Given the mixed monotone property of T,we know.

$T(p,q) \leq T(r,s)$ for $r \leq p, s \leq q$.

Now, since T is a mixed monotone mapping and $a_n \leq a_{n+1}$, $b_n \leq b_{n+1}$,by monotonicity of T.

We have:

$d(T(a_n,b_n),T(a_{n+1},b_{n+1})) \leq \frac{k}{2}[d(a_n,a_{n+1}) + d(b_n,b_{n+1})]$.

As $n \to \infty$, $d(a_n,a_{n+1})$ *and* $d(b_n,b_{n+1})$ both tend to zero, implying that $T(a_n,b_n)$ is a Cauchy sequences.

Theorem 1.2

If T is a Mixed monotone mapping satisfying the contractive condition, then there exists a unique pair $(a_0,b_0) \in X \times X$ such that $a_0 = T(a_0,b_0)$ *and* $b_0 = T(b_0,a_0)$.

Proof

We begin by considering the sequences $\{a_n\}$ and $\{b_n\}$ defined by:

$a_{n+1} = T(a_n,b_n), b_{n+1} = T(b_n,a_n)$.

Since T is mixed monotone, $a_{n+1} \leq a_n$ and $b_{n+1} \geq b_n$,leading to the conclusion that the sequences $\{a_n\}$ and $\{b_n\}$ are monotonic. By the properties of the complete metric space, these sequences converge to some limit point a_0 *and* b_0.

To show that $T(a_0,b_0) = a_0$ *and* $T(b_0,a_0,) = b_0$ consider:

$d(T(a_0,b_0),a_0) = \lim_{n\to\infty} d(T(a_n,b_n),\ a_{n+1} \ \leq k \lim_{n\to\infty} d(a_n,a_{n+1}) = 0$.

Similarly: $d(T(b_0,a_0),b_0) = \lim_{n\to\infty} d(T(b_n,a_n),\ b_{n+1} \ \leq k \lim_{n\to\infty} d(b_n,b_{n+1}) = 0$.

Hence, $T(a_0,b_0) = a_0$ *and* $T(b_0,a_0,) = b_0$ Uniqueness follows from the contractive condition and the completeness of X.

Lemma 1.2

Let $T: X \times X \to X$ be a mapping with the mixed monotone property in a partially ordered metric space (X,d). Satisfies the contraction $d(T(p,q),T(r,s)) \leq \frac{k}{2}[d(p,r) + d(q,s)]$,then for any two sequences $\{p_n\}$ and $\{q_n\}$ in X with $p_n \leq p_{n+1}$ and $q_n \leq q_{n+1}$,the sequence $\{T(p_n,q_n)$ converges.

Theorem 1.3

Let $(X,\leq)$ be a partially ordered set and (X,d) be a complete metric space if $T: X \times X \to X$ is a mixed monotone property and contraction condition, and if either condition (i) T *is continuous or* (ii)*The sequences* $\{a_n\}$ and $\{b_n\}$ are Cauchy, then T has a unique coupled fixed point (a,b) *in* X.

PRELIMINARIES

In this section we recall some definitions and examples.

Definition 2.1. [10] A Mapping $F: R \to R^+$ is called distribution function if it is non-decreasing and left-continuous with $in\ f_{x\in R}F(x) = 0$ is called a distance distribution function.

Definition 2.2. [10] A distance distribution function F satisfying $lim_{t\to\infty}F(t) = 1$ is called a Menger distance distribution function.

The set of all Menger distance distribution function is denoted by D^+.This space D^+ is partially ordered by the usual point wise ordering of functions, that is $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t\epsilon[0,\infty)$.

The maximal element for D^+ in this order is the distance distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, t = 0 \\ 1, t > 0 \end{cases}.$$

Definition 2.3. [10] A Triangular norm shortly known is as (t −norm) is a binary operation δ on [0,1] satisfying the following axioms:

T_1 : δ is associative and commutative.
T_2 : δ is continuous;
T_3 : $\delta(a, 1) = a, \forall\ a\ \epsilon\ [0,1]$
T_4 : $\delta(a, b) \leq \delta(c, d)\ , when\ a \leq c\ \&\ b \geq d\ \forall\ a, b, c, d\ \epsilon\ [0,1]$

Two typical examples of the continuous $t-$ norm is $\delta_P(a, b) = ab\ , \delta_M(a, b) = min\{a, b\}\ \forall\ a, b\epsilon[0,1]$

Now, the t −norm is recursively defined by $\delta^1 = \delta$ and

$\delta^n(a_1, a_2, a_3 \dots\dots\dots a_{n+1}) = \delta(\delta^{n-1}(a_1, a_2, a_3 \dots\dots\dots a_{n+1})).\forall\ n \geq 2$ And $a_i\epsilon[0,1]$

Where $i = 1,2,3 \dots\dots n + 1$. A t −norm δ is called Hadzic type if the family $\{\delta^n\}$ is equi-continuous at $a = 1$,that, for any ϵ (0,1),$\exists\ \delta \in$ (0,1) s.t

$a >$1-δ⇒ Δ^n(a) >1-ε , $\forall\ n \geq 1$.

δ_M is a trivial example of t −norm of Hadzic type [17].

Definition 2.4. [10] *Let* (X, F, t) be a Menger Probabilistic Metric space $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences s.t $a_n \to a\ , b_n \to b$, then $\lim_{n\to\infty} F_{a_n,b_n}$(t) =$F_{a,b}(t)$ for every continuity point of $F_{a,b}$.

Definition 2.5. [10] Let X be a non-empty set and T be a Mapping $T: X \times X \to X$ .an element $(a, b)\epsilon X \times X$ is said to be a coupled fixed point of T if $T(a, b) = a, T(b, a) = b$.

Definition 2.6. [10] Let (X, F, t) is a Menger Probabilistic Metric space where X be a non-empty set and consider two mapping $T: X \times X \to X$ and $h: X \times X \to X$.

1. An element $(a, b)\epsilon X \times X$ is said to be coupled coincidence point of h &T if $T(a, b) = h(a), T(b, a) = h(b)$.
2. 2. An element $(a, b)\epsilon X \times X$ is said to be coupled coincidence common fixed point of h &T if $T(a, b) = h(a) = a\ , T(b, a) = h(b) = b$.

Definition 2.7. [9] On consider two self -mapping f and g of a sub set of D of a menger metric space $((X, F, t)$ are called occasionally weakly compatible if $fx = gx$ and $fgx = gfx$. for some $x\epsilon D$.

Definition 2.8. [10] Let (X, F, t) is a Menger Probabilistic Metric space where X be a non-empty set and consider two mapping $T: X \times X \to X$ and $h: X \to X$ .The mappings T and h are said to be occasionally weakly compatible if there is a point $x\epsilon X$ which is a coupled coincidence point of f and g at which f and g commute.

MAIN RESULTS

Uniqueness of Fixed Point for Two Mapping in a Partially Menger Space

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Theorem 3.1

Let $(L, \leq)$ be a partially ordered set (Poset) equipped with a Menger space structure and let $f, g: X \to X$ be a two mapping. Suppose there exists a point x_0 in X such that $f(x_0) = g(x) = x_0$, $i.e$, x_0 is a fixed point of both f and g .If there exists another point x_1 in Xsuch that $f(x_1) = g(x_1) = x_1$, then $x_1 = x_0$.

Proof

Suppose x_0 and x_1 are both fixed point of f and g.Then , by the definition of a fixed point, we have two points.

1. $f(x_0) = g(x) = x_0$
2. $f(x_1) = g(x_1) = x_1$

On consider the partial ordering relation $\leq$ on since x_0 and x_1 are fixed points, we have.

$x_0 \leq f(x_0) = g(x)$
$x_1 \leq f(x_1) = g(x_1)$

By the properties of a Partially Menger space, we can establish the following:

1. If $x_0 \leq y$ and $x_1 \leq y$,then $d(x_0, y) = d(x_1, y)$.
2. If $x_0 \leq y$, then $f(x_0) \leq f(y)$ and $g(x_0) \leq y$. We have:
   $d(x_0, y) = d(x_1, y)$

This implies that $d(x_0, x_1) = 0$.since X is partially Menger space,

$d(x_0, x_1) = 0$ implies $x_0 = x_1$.

Therefore, if there exists more than one fixed point of f and g, they must coincide. Hence, the fixed point x_0 is unique in the partially Menger space $(X, \leq)$.

Example: Consider the partially ordered set $(X, \leq)$ defined as $X = \{a, b, c, d\}$ with the partial ordering relation $\leq$ given by the subset relation. That is

$a \leq b, a \leq c, a \leq d, b \leq c$ and $c \leq d$.

Let's define two mappings $f, g: X \to X$ as follows:

$f(a) = b, f(b) = c, f(c) = d, f(d) = a$
$g(a) = c, g(b) = d, g(c) = a, g(d) = b$

Verification of Fixed Points:

We can observe that:

For f, $f(a) = b$, $f(b) = c, f(c) = d$ and $f(d) = a$
For $g, g(a) = c, g(b) = d, g(c) = a,$ and $g(d) = b.$

APPLICATION OF THE THEOREM

According to the theorem, if there exists more than one fixed point for both f and g ,then they must coincide. However, in this example the fixed points of f and g are different ($i.e$ a and c) which satisfies the condition of the theorem. Therefore, the theorem does not apply in this case.

This example illustrates a scenario where the uniqueness of fixed points for two mappings in a partially Menger space does not hold.

CONCLUSION

In conclusion, this paper has elucidated the fundamental concepts of partial order Menger spaces, demonstrating their significance in mathematical analysis. Through rigorous examination of properties and theorems, we have advanced understanding of these spaces. Further research in this area holds promise for applications in diverse fields, underscoring the importance of continued exploration and development.

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