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Comparative Study of Uniform and Non-Uniform Beams Resting on Bi-Parametric Foundation Under Harmonic Load with Time-Dependent Boundary Conditions

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Abstract

In this study, we investigated uniform and non-uniform damped and time-dependent elastic beams resting on a bi-parametric foundation subjected to a harmonically varying moving load. The beam properties, including moment of inertia and mass per unit length, were assumed to be either constant or varying with position along the beam span in this study. The solution methods were based on the Mindlin-Goodman method, Fourier sine transformation, weighted residual method, integral transformation, and the theory of convolution. The response amplitudes of the beams were calculated by considering the structural parameters in the models' equations with the aid of Maple software, and the results were presented graphically. It is seen from the graphs that, as values of the damping coefficient and axial force increase, the response amplitudes of both uniform and non-uniform beams decrease. Also, decreases in deflection profiles of both uniform and non-uniform beams were brought about as a result of increases in foundation modulus and shear modulus. Increases in the values of load natural frequency and velocity of the moving load led to decreases in the response amplitude of uniform and non-uniform beams. The findings also revealed that, in each effect of the structural parameters considered, it is seen that, the deflection profiles of the uniform beam is significantly higher than the non-uniform beam, that is, for constant values of all the structural parameters, the response amplitudes of uniform beam is much higher than that of non-uniform beam and as a result, resonance effect reach earlier in the uniform beam than the non-uniform beam.

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INTRODUCTION

Investigations into dynamical structures, such as roads, bridges, and rails, subjected to moving loads, such as trains, cars, ships, and lorries, have been the concern of many researchers in the fields of applied mathematics, mathematical physics, and engineering. High vibrations of these structures, especially bridges, as a result of high-speed and heavy vehicles, can seriously affect the internal stress of the bridges. To design and operate structures effectively, it is imperative to have a better understanding of their dynamic properties. The earlier work in this field of study is the work of Stokes [1] and Michaltsos [2], who considered a girder with mass negligible compared with that of the mass of moving load of constant magnitude. Much later, a moving load whose mass was considered insignificant compared to the masses of the girders was investigated by Krylov, Timoshenko, and Lowan [3–5].

Several researchers have found that the properties of a beam, such as the moment of inertia and mass per unit length, significantly affect the response amplitudes of dynamic systems, such as bridges. They investigated a situation in which those properties were considered to be constant with position x along the span L of the beam. For example, Abbas et al. [6] considered moving loads on Timoshenko and Bernoulli-Euler uniform beams resting on elastic foundations using the differential quadrature method. Their findings revealed that the Timoshenko beam produced a higher deflection profile than the Bernoulli-Euler beam because of the shear effects in the Timoshenko beam. Oyelade and Obanishola [7] used a numerical approach to investigate the effect of shear on the wave number and damping of a uniform beam with elastic and metal material foundations. Dynamic analysis of a uniform beam resting on an elastic foundation using Fourier transformation was explored by Michael and Sefa [8], and Arathy and Bennet [9] used the finite element method to analyze a uniform beam resting on a Winkler foundation because of its simplicity. Their results showed that increases in the Young's modulus, soil capacity, beam depth, and beam width reduce the response amplitude of the beam. Mahmut et al. [10] used the Stokes transformation and Fourier series to investigate a uniform beam of steel fiber-reinforced self-compacting concrete on an elastic foundation. They concluded that noticeable effects were observed regarding fiber reinforcement on the dynamic response of the beam.

However, the properties of the beam may vary with position x along the span L of the beam, and when this occurs, it makes the dynamical system cumbersome to handle because the governing equations describing such a dynamical system will then have both singular and variable coefficients. In view of this, Adedowole and Famuagun [11] considered a non-uniform prismatic beam on a variable elastic foundation under constant and variable harmonic magnitudes of moving loads. They concluded that the constant-magnitude moving load produced less deflection than the variable-harmonic-magnitude moving load. The influence of a variable axial force on a non-uniform thick beam subjected to a distributed load was explored by Jimoh et al. [12]. They obtained a closed-form solution of the dynamical system using Galerkin's techniques with the series representation of the Heaviside function, Laplace transformation alongside the convolution theorem, and Stubble's asymptotic methods.

The response of a non-uniform simply supported Rayleigh beam under the action of a moving distributed load was explored by Oni and Ayankop [13]. Abdelghany [14] investigated the dynamic behavior of a non-uniform beam resting on a viscoelastic foundation under a moving load. The free vibration analysis of a non-uniform Rayleigh beam resting on a variable Winkler elastic foundation using the differential transform method was examined by Olotu et al. [15].

The elastic foundation on which the structures rest also has a significant impact on the dynamic response of the dynamical system. Some researchers [1–15] considered a one-parameter foundation model; however, because of the shortcomings of the one-parameter foundation, most other studies explored two-parameter foundations in their works. The work of Jimoh et al. [16] investigated the effect of two parameters on the foundation of a clamped-clamped elastic beam subjected to a concentrated load. They obtained their solution using the generalized integral transformation method with a series representation of the Dirac delta function, the asymptotic method, the integral transformation, and the convolution theorem. They concluded that the effect of the Pasternak foundation parameter is more noticeable than that of the Winkler foundation parameter. The dynamic behavior of a damped shear beam on a bi-parametric foundation was examined by Ajijola [17]. He established the resonance phenomenon in a dynamic system and the speed at which it occurs. Hammed et al. [18] used Fourier series and differential transformations to examine the response amplitudes of an elastic beam on a Pasternak foundation subjected to a distributed moving load. They concluded that the amplitudes of both the upper and lower beams increased with the speed of the moving load. Lucos et al. [19] obtained the response amplitudes of an axially loaded Euler-Bernoulli beam resting on two-parameter foundations using the finite element and Rayleigh-Ritz methods. They concluded that while the foundation parameter increases the frequency parameters, the axial force decreases. Ahmed et al. [20] used the Moove-Gibson-Thompson model to analyze the displacement, temperature, deflection, and flexure moment of a viscoelastic beam resting on a Pasternak foundation. Some of the other researchers who considered two parameters foundation in their studies are: Wallison et al. [21], Awodola [22], Baran [23], Saurabh [24], Jimoh [25], to mention but a few.

It is important to note that all the aforementioned researchers considered only the classical boundary conditions, which are simply supported, clamped-clamped, and clamped-free boundary conditions. The non-classical boundary conditions, which include elastically supported and time-dependent boundary conditions, are very scanty in the literature. Few researchers have considered non-classical boundary conditions in their studies, including Mindlin and Goodman [26], Oni and Awodola [27], Ajijola [28], Bayem et al. [29], and Adedowole [30].

After a literature survey, it can be seen that the comparison of uniform and non-uniform beams subjected to harmonic loads on two parameters with non-classical boundary conditions remains outstanding. Hence, this study examined the dynamic analysis of damped uniform and non-uniform beams resting on a bi-parametric foundation and subjected to harmonically varying moving loads with time-dependent boundary conditions.

THE GOVERNING EQUATIONS

The governing partial differential equations describing the uniform and non-uniform beams on a bi-parametric foundation and traversed by a harmonic moving load of mass m moving with velocity c and damping term included are given by Frybal [31].

$$EJ\frac{\partial^4\psi(x,t)}{\partial x^4}+\overline{W}\frac{\partial^2\psi(x,t)}{\partial t^2}-N\frac{\partial^2\psi(x,t)}{\partial x^2}+\alpha\frac{\partial\psi(x,t)}{\partial t}+K\psi(x,t)-G\frac{\partial^2\psi(x,t)}{\partial x^2}$$
$$=Pcos\omega t\,\delta(x-vt) \tag{1}$$

and

$$\frac{\partial^2}{\partial x^2}\left(EJ(x)\frac{\partial^2\psi(x,t)}{\partial x^2}\right)+\overline{W}(x)\frac{\partial^2\psi(x,t)}{\partial t^2}-N\frac{\partial^2\psi(x,t)}{\partial x^2}+\varepsilon\frac{\partial\psi(x,t)}{\partial t}-G\frac{\partial^2\psi(x,t)}{\partial x^2}+K\psi(x,t)$$
$$=Pcos\omega t\,\delta(x-vt) \tag{2}$$

Where,

EJ is the flexural rigidity of the structures, $EJ(x)=$ the variable flexural rigidity of the structures, $\overline{W}$= the mass per unit length of the beam, $\overline{W}(x)$= variable mass per unit length of the beam.

K is the foundation modulus, G = Shear modulus, N is the axial force, $Alpha\ (\alpha)$ the damping coefficient, E is the Young's modulus, J is the constant moment of inertia, $J(x)$=variable moment of inertia, $\psi(x,t)$= transverse displacement, *x* is the spatial coordinate, *t* is the time coordinate, $Omega\ (\omega\ )$ is the circular frequency of the moving load. v = velocity of the moving load.

$$D_i[\psi(0,t)]=f_i(t), i=1,2 \tag{3}$$
$$D_i[\psi(L,t)]=f_i(t), i=1,2 \tag{4}$$

Where, D_i are differential operators of order less than or equal to three.

For example, if the beam in question has a simple support at both ends, then

$$D_1=1, D_2=\frac{\partial^2}{\partial x^2}, D_3=1\ D_4=\frac{\partial^2}{\partial x^2} \tag{5}$$

The initial conditions of the motion are specified by the functions.

$$\psi(x,0)=0=\frac{\partial\psi(x,0)}{\partial t} \tag{6}$$

The variable mass per unit length of the beam and the variable moment of inertia are given as

$$\bar{W}(x) = \bar{W}_o\left(1 + Sin\frac{\pi x}{L}\right) \tag{7}$$

and

$$J(x) = J_o\left(1 + Sin\frac{\pi x}{L}\right)^3 \tag{8}$$

Where $\bar{W}_o$ and J_o are the constant mass per unit length and constant moment of inertia of the non-uniform beam, respectively.

Substituting Equations (7) and (8) into Equation (2), after simplification and rearrangement, Equation (2) becomes

$$\begin{aligned}
&\frac{EJ_o}{4}\left[\frac{\pi^2}{L^2}\left(-15Sin\frac{\pi x}{L} + 9Sin\frac{3\pi x}{L} + 24Cos\frac{2\pi x}{L}\right)\frac{\partial^2\psi(x,t)}{\partial x^2}\right]\\
&+\frac{2\pi}{L^2}\left(15\,Cos\frac{\pi x}{L} - 3Cos\frac{3\pi x}{L} + 12Sin\frac{2\pi x}{L}\right)\frac{\partial^3\psi(x,t)}{\partial x^3}\\
&+\left(10 + 15Sin\frac{\pi x}{L} - Sin\frac{3\pi x}{L} - 6Cos\frac{2\pi x}{L}\right)\frac{\partial^4\psi(x,t)}{\partial x^4}\\
&+\bar{W}_o\left(1 + Sin\frac{\pi x}{L}\right)\frac{\partial^2\psi(x,t)}{\partial t^2} + \varepsilon\frac{\partial\psi(x,t)}{\partial t} - (N+G)\frac{\partial^2\psi(x,t)}{\partial x^2}\\
&+K\psi(x,t) = Pcoswt\,\delta(x - vt)
\end{aligned} \tag{9}$$

METHODS OF SOLUTION

To obtain solutions to Equations (1) and (9), the approach of Mindlin and Goodman was extended, and we proceeded as follows:

An auxiliary variable $\varphi(x,t)$ in the form

$$\psi(x,t) = \varphi(x,t) + \sum_{i=0}^{4} f_i(t)\,g_i(x) \tag{10}$$

is introduced.

Substituting Equation (10) into the boundary value problems in Equations (1), (3), (4), and (9) transform Equations (1) and (9) into boundary value problems in terms of $\varphi(x,t)$. The functions $g_i(x)$ are chosen to render the boundary conditions for the boundary value problems in $\varphi(x,t)$ homogeneous.

Substituting Equations (10) into Equations (1) and (9), after simplification, yields

$$\begin{aligned}
&EJ\frac{\partial^4\varphi(x,t)}{\partial x^4} + \bar{W}\frac{\partial^2\varphi(x,t)}{\partial t^2} - N\frac{\partial^2\varphi(x,t)}{\partial x^2} + \alpha\frac{\partial\varphi(x,t)}{\partial t} + K\varphi(x,t) - G\frac{\partial^2\varphi(x,t)}{\partial x^2}\\
&Qcos\omega t\delta(x - vt) - \sum_{i=1}^{4}\left\{\begin{array}{c} f_i(t)EI\frac{\partial^4 g_i(x)}{\partial x^4} + \mu g_i(x)\ddot{f}_i(t) + g_i(x)\dot{f}_i(t) \\ f_i(t)N\frac{\partial^2 g_i(x)}{\partial x^2} + f_i(t)Kg_i(x) - f_i(t)\frac{\partial^2 g_i(x)}{\partial x^2} \end{array}\right\}
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\frac{EI_o}{4\bar{W}_o}\left(10 + 15Sin\frac{\pi x}{L} - Sin\frac{3\pi x}{L} - 6Cos\frac{2\pi x}{L}\right)\frac{\partial^4\varphi(x,t)}{\partial x^4}\\
&+\frac{EI_o}{2L\bar{W}_o}\left(15\,Cos\frac{\pi x}{L} - 3Cos\frac{3\pi x}{L} + 12Sin\frac{2\pi x}{L}\right)\frac{\partial^3\varphi(x,t)}{\partial x^3}\\
&+\left[\frac{EI_o}{4\bar{W}_o}\frac{\pi^2}{L^2}\left(-15Sin\frac{\pi x}{L} + 9Sin\frac{3\pi x}{L} + 24Cos\frac{2\pi x}{L}\right) - (N+G)\right]\frac{\partial^2\varphi(x,t)}{\partial x^2}\\
&+\left(1 + Sin\frac{\pi x}{L}\right)\frac{\partial^2\varphi(x,t)}{\partial t^2} + \frac{\varepsilon}{\mu_o}\frac{\partial\varphi(x,t)}{\partial t} + \frac{K}{\mu_o}\varphi(x,t)\\
&= Qcoswt\,\delta(x - vt)
\end{aligned}$$

$$-\sum_{i=1}^{4}\left[\begin{array}{c}\frac{EJ_o}{4\bar{W}_o}\left(10+15Sin\frac{\pi x}{L}-Sin\frac{3\pi x}{L}-6Cos\frac{2\pi x}{L}\right)\frac{\partial^4 f_i(t)g_i(x)}{\partial x^4}\\ +\frac{EJ_o}{2L\bar{W}_o}\left(15\,Cos\frac{\pi x}{L}-3Cos\frac{3\pi x}{L}+12Sin\frac{2\pi x}{L}\right)\frac{\partial^3 f_i(t)g_i(x)}{\partial x^3}\\ +\left(\frac{EJ_o}{4\bar{W}_o}\frac{\pi^2}{L^2}\left(-15Sin\frac{\pi x}{L}+9Sin\frac{3\pi x}{L}+24Cos\frac{2\pi x}{L}\right)-(N+G)\right)\frac{\partial^2 f_i(t)g_i(x)}{\partial x^2}\\ +\left(1+Sin\frac{\pi x}{L}\right)\frac{\partial^2 f_i(t)g_i(x)}{\partial t^2}+\frac{\varepsilon}{\mu_o}\frac{\partial f_i(t)g_i(x)}{\partial t}+\frac{K}{\mu_o}f_i(t)g_i(x)\end{array}\right] \quad (12)$$

Where,

$$Q = \mathrm{Mg} \quad (13)$$

The assumed expression in Equation (10) must satisfy the boundary conditions in Equations (3, 4).

Thus,

$$D_i[\varphi(0,t)] + \sum_{j=0}^{4} f_i(t)\, D_i[g_i(0)] = f_i(t),\ i = 1,2 \quad (14)$$

$$D_i[\varphi(L,t)] + \sum_{j=0}^{4} f_i(t)\, D_i[g_i(L)] = f_i(t),\ i = 3, \quad (15)$$

Substituting Equation (10) into the initial conditions in Equation (6) to obtain

$$\varphi(x,0) = -\sum_{i=0}^{4} f_i(0)\, g_i(x) \quad (16)$$

$$\frac{\partial\varphi(x,0)}{\partial t} = -\sum_{i=0}^{4} f_i(0)\, g_i(x) \quad (17)$$

Using the Mindlin-Goodman method, the boundary conditions in Equations (14) and (15) in terms of can be made homogeneous if the function is chosen such that the conditions given by

$$D_i\left[g_j(0)\right] = \delta_{ij} (i = 1,2\ j = 1,2,3,4) \quad (18)$$

$$D_i\left[g_j(L)\right] = \delta_{ij} (i = 1,2\ j = 1,2,3,4) \quad (19)$$

are satisfied.

Where,

$$\delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} \quad (20)$$

Using Equations (18) and (19) in the non-homogeneous boundary conditions in Equations (14) and (15) to obtain the homogeneous boundary conditions for Equations (1) and (9), we have

$$D_i[\varphi(0,t)] = 0, i = 1,2 \quad (21)$$

$$D_i[\varphi(L,t)] = 0, i = 1,2 \quad (22)$$

The original Equations (1) and (2) are now reduced to solving the non-homogeneous partial differential Equations (11) and (12) subject to the homogeneous boundary conditions in Equations (21) and (22) with initial conditions in Equations (16) and (17).

To solve the initial boundary value problem of Equations (11), (21), and (22), a Fourier integral transform is defined by

$$\varphi(m,t) = \int_0^L \varphi(x,t) \sin\frac{m\pi x}{L}\, dx, m = 1,2,3 \quad (23)$$

With the inverse

$$\varphi(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\varphi(m,t)\sin\frac{m\pi x}{L} \tag{24}$$

is employed.

Substituting Equation (23) into Equation (11) after some simplification and rearrangement to obtain

$$\bar{W}\varphi_{tt}(m,t) + \alpha\varphi_t(m,t) + \left(\alpha_a^2 + \alpha_b^2 + \alpha_c^2 + K\right)\varphi(x,t) = Q\cos\omega t\,\delta(x-ct)$$
$$- \left[Q_a(t) + Q_b(t) + Q_c(t) + Q_d(t) + Q_e(t) + Q_f(t)\right] \tag{25}$$

Where,

α_a^2, α_b^2, α_c^2, $Q_a(t)$, $Q_c(t)$, $Q_d(t)$, $Q_e(t)$, $Q_f(t)$, A_1, A_2, A_3, N_1, N_2, N_3, I_1, I_2 are given by Jimoh et al. [31].

By evaluating the integrals and substituting the results into Equation (25), after some simplification and rearrangement, we obtain

$$\varphi_{tt}(m,t) + A_1\varphi_t(m,t) + A_2\varphi(m,t) = \frac{Q}{\bar{W}}\cos\omega t\,Sin\frac{m\pi ct}{L} - (N_1 + N_2 + N_3) \tag{26}$$

Equation (26) is the transformed equation of a uniform damped beam on a bi-parametric foundation subjected to a harmonic moving load with time-dependent boundary conditions.

Analysis of the Transformed Equation

Following the procedure by Jimoh et al. [31], Equation (26) can be transformed into

$$\varphi_{tt}(m,t) + A_1\varphi_t(m,t) + A_2\varphi(m,t) = \frac{Q}{\bar{W}}\cos\omega t\,Sin\frac{m\pi ct}{L}$$
$$-\left\{\begin{matrix} -C_{f1}\Omega^2 Sin\Omega t + C_{f2}\left(\beta^2 e^{-\beta t}Sin\Omega t - 2\beta\Omega e^{-\beta t}Cos\Omega t - \Omega^2 e^{-\beta t}Sin\Omega t\right) \\ +A_1\left[C_{f1}\Omega Cos\Omega t + C_{f2}\left(\Omega e^{-\beta t}Cos\Omega t - \beta e^{-\beta t}Sin\Omega t\right)\right] \\ A_3\left(C_{f1}Sin\Omega t + C_{f2}e^{-\beta t}Sin\Omega t\right) \end{matrix}\right\} \tag{27}$$

Taking the Laplace transform of Equation (27) defined by

$$F(s) = \int_0^L e^{-st} f(t)dt \tag{28}$$

In conjunction with the initial conditions in Equations (16) and (17), after some simplification and rearrangement, Equation (27) becomes

$$\bar{\varphi}(m,s) = \frac{1}{r_1 - r_2}\left[A_4\left(\frac{\gamma_1}{S^2+\gamma_1^2} - \frac{\gamma_2}{S^2+\gamma_2^2}\right) + A_5\frac{\Omega}{S^2+\Omega^2} + A_6\frac{\Omega}{(s+\Omega)^2+\Omega^2} + A_7\frac{s+\beta}{(s+\beta)^2+\Omega^2} + A_8\frac{s}{S^2+\Omega^2}\right]\mathrm{x}\left[\frac{1}{s+r_2} - \frac{1}{s+r_1}\right] \tag{29}$$

Where,

$$\left.\begin{matrix} A_4 = \frac{\mathrm{Mg}}{2\mu}, A_5 = C_{f1}\left(\Omega^2 - A_3\right), A_6 = -C_{f2}\left(\Omega^2\beta^2 - A_3\right) + A_1\beta \\ A_7 = C_{f2}(2\beta\Omega - A_1), A_8 = -A_1 Cf_1\Omega, C_{f1} = I_1 - \frac{I_2}{L}, C_{f2} = \frac{I_2}{L} \\ r_1 = -\frac{1}{2}\left(A_1 + \sqrt{A_1^2 - 4A_1}\right)\ r_2 = -\frac{1}{2}\left(A_1 - \sqrt{A_1^2 - 4A_1}\right) \\ \gamma_1 = \omega + \frac{m\pi v}{L}\ \gamma_2 = \omega - \frac{m\pi v}{L} \end{matrix}\right\} \tag{30}$$

To obtain the Laplace inversion of Equation (29), we use the convolution integral defined as

$$f(t) * g(t) = \int_0^L f(u)g(t-u)du \tag{31}$$

So that the inversion of Equation (29) can be simplified as

$$U(m,t) = \emptyset \begin{bmatrix} A_4(R_1 - R_2) + A_5(R_3 - R_9) + A_6(R_4 + R_{10}) + A_8(R_6 - R_{12}) \\ +A_7(R_5 - R_{11}) + A_8(R_6 - R_{12}) - R_7 + R_7 \end{bmatrix} \tag{32}$$

Where,

$(R_1 - \ R_{12})$ are

$$\left.\begin{array}{c} R_1 = \int_0^t e^{-r_2(t-u)}\, Sin\, \gamma_1 u du \; R_2 = \int_0^t e^{-r_2(t-u)}\, Sin\, \gamma_2 u du \\ R_3 = \int_0^t e^{-r_2(t-u)}\, Sin\, \Omega u du \; R_4 = \int_0^t e^{-r_2(t-u+\Omega u)}\, Sin\, \Omega u du \\ R_5 = \int_0^t e^{-r_2(t-u+\beta u)}\, Cos\, \Omega u du \; R_6 = \int_0^t e^{-r_2(t-u)}\, Cos\, \Omega u du \\ R_7 = \int_0^t e^{-r_1(t-u)}\, Sin\, \gamma_1 u du \; R_8 = \int_0^t e^{-r_1(t-u)}\, Sin\, \gamma_1 u du \\ R_9 = \int_0^t e^{-r_1(t-u)}\, Sin\, \Omega u du \; R_{10} = \int_0^t e^{-r_1(t-u+\Omega u)}\, Sin\, \Omega u du \\ R_{11} = \int_0^t e^{-r_1(t-u+\beta u)}\, Cos\, \Omega u du \; R_{12} = \int_0^t e^{-r_1(t-u)}\, Cos\, \Omega u du \end{array}\right\} \tag{33}$$

Putting Equation (32) into Equation (24) to obtain

$$\varphi(x,t) = \frac{2}{L}\sum\sum_{m=1}^{\infty} \emptyset \begin{bmatrix} A_4(R_1 - R_2) + A_5(R_3 - R_9) + A_6(R_4 - R_{10}) + A_8(R_6 - R_{12}) \\ +A_7(R_5 - R_{11}) + A_8(R_6 - R_{12}) - R_7 + R_7 \end{bmatrix} \sin\frac{m\pi x}{L} \tag{34}$$

By evaluating the integrals and substituting the results into Equation (34) to obtain

$(x,t)\ \varphi(x,t) =$

$$\frac{2}{L}\sum_{m=1}^{\infty} \emptyset \left\{ \begin{array}{c} A_4 \begin{bmatrix} \frac{\gamma_1}{\gamma_1^2 + r_2^2}\left(e^{\,-r_2 t} - Cos\gamma_1 t + \frac{r_2}{\gamma_1} Sin\gamma_1 t\right) \\ -\frac{\gamma_2}{\gamma_2^2 + r_2^2}\left(e^{\,-r_2 t} - Cos\gamma_2 t + \frac{r_2}{\gamma_2} Sin\gamma_2 t\right) \end{bmatrix} \\ +A_5 \begin{bmatrix} \frac{\gamma_2}{\Omega^2 + r_2^2}\left(e^{\,-r_2 t} - Cos\Omega t + \frac{r_2}{\Omega} Sin\Omega t\right) \\ -\frac{\Omega}{\Omega^2 + r_1^2}\left(e^{\,-r_1 t} - Cos\Omega t + \frac{r_1}{\Omega} Sin\Omega t\right) \end{bmatrix} \\ +A_6 \begin{bmatrix} \frac{\Omega}{\Omega^2 + r_2^2(1-\Omega)^2}\left(e^{\,-r_2\Omega t}\, Cos\Omega t - e^{\,-r_2 t} + e^{r_2\Omega t}\frac{r_2(1-\Omega)}{\Omega} Sin\Omega t\right) \\ -\frac{\Omega}{\Omega^2 + r_1^2(1-\Omega)^2}\left(e^{r_1\Omega t} Cos\Omega t - e^{r_1 t} + \frac{r_1(1-\Omega)}{\Omega}\right) e^{r_1\Omega t} Sin\Omega t \end{bmatrix} \\ +A_7 \begin{bmatrix} \frac{\Omega}{\Omega^2 + r_2^2(1-\beta)^2}\left(e^{r_2\beta t} + \frac{r_2(1-\beta)}{\Omega}\left(e^{r_2\beta t} Cos\Omega t - e^{\,-r_2 t}\right)\right) \\ -\frac{\Omega}{\Omega^2 + r_1^2(1-\beta)^2}\left(e^{r_1\beta t} + \frac{r_1(1-\beta)}{\Omega}\left(e^{r_1\beta t} Cos\Omega t - e^{r_1 t}\right)\right) \end{bmatrix} \\ +A_8 \begin{bmatrix} \frac{1}{\Omega + r_2 - \beta}\left(e^{\,-\beta t}\, Sin\,\Omega t + \frac{(r_2-\beta)e^{-\beta t}\, Cos\,\Omega t}{\Omega}\right) \\ -\frac{1}{\Omega + r_1 - \beta}\left(e^{\,-\beta t}\, Sin\,\Omega t + \frac{(r_1-\beta)e^{-\beta t}\, Cos\,\Omega t}{\Omega}\right) \end{bmatrix} \\ -\frac{\gamma_1}{\gamma_1^2 + r_1^2}\left(e^{\,-r_1 t} - Cos\gamma_1 t + \frac{r_1}{\gamma_1} Sin\gamma_1 t\right) + \frac{\gamma_2}{\gamma_2^2 + r_1^2}\left(e^{\,-r_1 t} - Cos\gamma_2 t + \frac{r_1}{\gamma_2} Sin\gamma_2 t\right) \end{array} \right\} Sin\frac{m\pi x}{L} \tag{35}$$

Equation (35) represents the response amplitude of the damped uniform beam resting on a bi-parametric foundation and subjected to a harmonically varying moving load with time-dependent boundary conditions.

Similarly, to obtain the solution to the initial boundary value problem in Equations (12), (21), and (22) above, we employed Galerkin's method defined by

$$\varphi(x\ t) = \sum_{m=1}^{n} Y_m(t)\ \varphi_m(x)\ \ m = 1,2 \tag{36}$$

Where, $\varphi_m(x)$. Given below is chosen as a suitable kernel of the Galerkin method such that the given boundary conditions are satisfied.

$$\varphi_m(x) = Sin\frac{m\pi x}{L} \tag{37}$$

Following the same procedures as Jimoh et al. [31], Equation (12) can be transformed into

$$\ddot{Y}_m(t) + \alpha_1\dot{Y}_m(t) + \alpha_2 Y_m(t)$$
$$= \frac{P}{D_1\bar{W}_o} coswt\, Sin\theta_c t + A_9 Sin\Omega t + A_{10}e^{-\beta t}Sin\Omega t + A_{11}e^{-\beta t}Cos\Omega t \tag{38}$$

Where,

α_1, α_2, θ_c, A_9, A_{10}, A_{11}, D_1, β, and Ω are defined by Jimoh et al. [32], and $(\cdot)$ is the derivative with respect to time t.

Equation (38) is the transformed equation of a non-uniform beam resting on a bi-parametric foundation under a harmonic load with time-dependent boundary conditions.

To obtain its solution, Equation (38) is subjected to Equations (28) and (31). After simplification and rearrangement, Equation (38) becomes

$$Y_m(t) = \emptyset\left[(I_1 - I_6) + \left(I_2 - I_7\right) + (I_3 - I_8) + (I_4 - I_9) + (I_5 - I_{10})\right)\right] \tag{39}$$

Where,

$$\emptyset = \frac{1}{r_1 + r_2} \tag{40}$$

$$\left.\begin{array}{c}
I_1 = -\frac{R}{2}\int_0^t Sin(w+\theta_c)u\, e^{-r_2(t-u)}\, du\, , I_2 = -\frac{R}{2}\int_0^t Sin(w+\theta_c)u\, e^{-r_1(t-u)}\, du \\
I_3 = A_9\int_0^t Sin\,\Omega u\, e^{-r_2(t-u)}\, du\, , I_4 = A_{10}\int_0^t Sin\,\Omega u\, e^{-(r_2-\beta)u - r_2 t}\, du \\
I_5 = A_{11}\int_0^t Cos\,\Omega u\, e^{-(r_2-\beta)u - r_2 t}\, du\, , I_6 = \frac{R}{2}\int_0^t Sin(w+\theta_c)u\, e^{-r_1(t-u)}\, du \\
I_7 = -\frac{R}{2}\int_0^t Sin(w-\theta_c)u\, e^{-r_1(t-u)}\, du\, , I_8 = A_9\int_0^t Sin\,\Omega u\, e^{-r_1(t-u)}\, du \\
I_9 = A_{10}\int_0^t Sin\,\Omega u\, e^{-(r_1-\beta)u - r_1 t}\, du\, , I_{10} = A_{10}\int_0^t Cos\,\Omega u\, e^{-(r_1-\beta)u - r_1 t}\, du
\end{array}\right\} \tag{41}$$

By simplifying the integrals in Equation (41) and substituting the results into Equation (39), we obtain

$$Y_m(t) = \emptyset\left\{\begin{array}{c}
\frac{R}{2}\left[\begin{array}{l}\frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_2 t} - Cos\,(w-\theta_c)t + \frac{r_2\,Sin(w-\theta_c)t}{w-\theta_c}\right) \\ -\frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_2 t} - Cos\,(w-\theta_c)t + \frac{r_2\,Sin(w-\theta_c)t}{w-\theta_c}\right)\end{array}\right] \\
-\frac{R}{2}\left[\begin{array}{l}\frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_1 t} - Cos\,(w-\theta_c)t + \frac{r_1\,Sin(w-\theta_c)t}{w-\theta_c}\right) \\ -\frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_1 t}\, Cos\,(w-\theta_c)t + \frac{r_1\,Sin(w-\theta_c)t}{w-\theta_c}\right)\end{array}\right] \\
+A_9\left[\begin{array}{l}\frac{\Omega}{\Omega+r_2^2}\left(e^{-r_2 t} - Cos\,\Omega t + \frac{r_2\,Sin\,\Omega t}{\Omega}\right) \\ -\frac{\Omega}{\Omega+r_1^2}\left(e^{-r_1 t} - Cos\,\Omega t + \frac{r_1\,Sin\Omega t}{\Omega}\right)\end{array}\right] \\
+A_{10}\left[\begin{array}{l}\frac{1}{\Omega+r_2-\beta}\left(e^{-r_2 t} - e^{-\beta t}\,Cos\,\Omega t + \frac{(r_2-\beta)e^{-\beta t}\,Sin\,\Omega t}{\Omega}\right) \\ -\frac{1}{\Omega+r_1-\beta}\left(e^{-r_1 t} - e^{-\beta t}\,Cos\,\Omega t + \frac{(r_1-\beta)e^{-\beta t}\,Sin\,\Omega t}{\Omega}\right)\end{array}\right] \\
+A_{11}\left[\begin{array}{l}\frac{1}{\Omega+r_2-\beta}\left(e^{-\beta t}\,Sin\,\Omega t + \frac{(r_2-\beta)e^{-\beta t}\,Cos\,\Omega t}{\Omega}\right) \\ \frac{1}{\Omega+r_1-\beta}\left(e^{-\beta t}\,Sin\,\Omega t + \frac{(r_1-\beta)e^{-\beta t}\,Cos\,\Omega t}{\Omega}\right)\end{array}\right]
\end{array}\right\} \tag{41}$$

Substituting Equations (37) and (42) into Equation (36) yields

$$
= \sum_{m=1}^{n} \emptyset \left\{ \begin{array}{c}
\frac{R}{2}\left[\begin{array}{l} \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_2 t} - Cos\,(w-\theta_c)t + \frac{r_2\, Sin(w-\theta_c)t}{w-\theta_c}\right) \\ -\frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_2 t} - Cos\,(w-\theta_c)t + \frac{r_2\, Sin(w-\theta_c)t}{w-\theta_c}\right) \end{array}\right] \\
-\frac{R}{2}\left[\begin{array}{l} \frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_1 t} - Cos\,(w-\theta_c)t + \frac{r_1\, Sin(w-\theta_c)t}{w-\theta_c}\right) \\ -\frac{w-\theta_c}{(w-\theta_c)^2+r_1^2}\left(e^{-r_1 t}\, Cos\,(w-\theta_c)t + \frac{r_1\, Sin(w-\theta_c)t}{w-\theta_c}\right) \end{array}\right] \\
+A_9\left[\begin{array}{l} \frac{\Omega}{\Omega+r_2^2}\left(e^{-r_2 t} - Cos\,\Omega t + \frac{r_2\, Sin\,\Omega t}{\Omega}\right) \\ -\frac{\Omega}{\Omega+r_1^2}\left(e^{-r_1 t} - Cos\,\Omega t + \frac{r_1\, Sin\Omega t}{\Omega}\right) \end{array}\right] \\
+A_{10}\left[\begin{array}{l} \frac{1}{\Omega+r_2-\beta}\left(e^{-r_2 t} - e^{-\beta t}\, Cos\,\Omega t + \frac{(r_2-\beta)e^{-\beta t}\, Sin\,\Omega t}{\Omega}\right) \\ -\frac{1}{\Omega+r_1-\beta}\left(e^{-r_1 t} - e^{-\beta t}\, Cos\,\Omega t + \frac{(r_1-\beta)e^{-\beta t}\, Sin\,\Omega t}{\Omega}\right) \end{array}\right] \\
+A_{11}\left[\begin{array}{l} \frac{1}{\Omega+r_2-\beta}\left(e^{-\beta t}\, Sin\,\Omega t + \frac{(r_2-\beta)e^{-\beta t}\, Cos\,\Omega t}{\Omega}\right) \\ -\frac{1}{\Omega+r_1-\beta}\left(e^{-\beta t}\, Sin\,\Omega t + \frac{(r_1-\beta)e^{-\beta t}\, Cos\,\Omega t}{\Omega}\right) \end{array}\right]
\end{array} \right\} Sin\frac{m\pi x}{L} \qquad (43)
$$

Equation (43) represents the transverse displacement of a non-uniform elastic beam resting on a bi-parametric foundation and subjected to a harmonic moving load with time-dependent boundary conditions.

DISCUSSION OF RESULTS

Figure 1(a) and (b) shows the response amplitudes of uniform and non-uniform beams for various values of the foundation modulus, respectively. The graphs show that as the values of the foundation modulus increase, the response amplitude of the uniform beam decreases significantly, whereas that of the non-uniform beam decreases slightly.

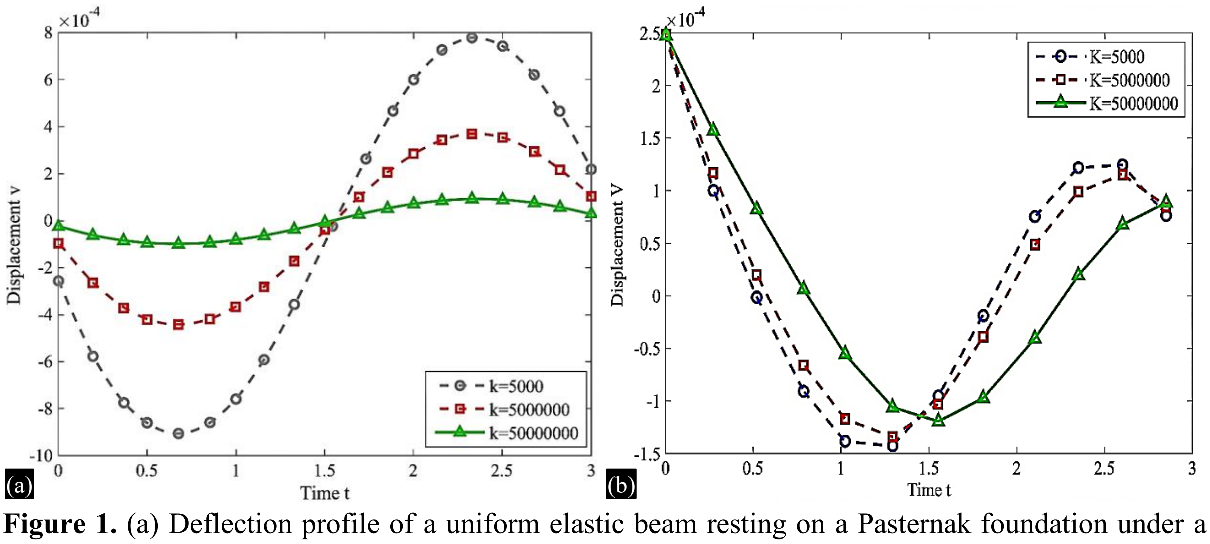

Figure 1. (a) Deflection profile of a uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of foundation modulus (K) with time-dependent boundary conditions; (b) Deflection profile of a non-uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of foundation modulus (K) with time-dependent boundary conditions.

Figure 2(a) and (b) depicts the response amplitudes of uniform and non-uniform beams, respectively, for different values of the damping coefficient. The graphs show that as the values of the damping coefficient increase, the response amplitudes of both uniform and non-uniform beams decrease. The deflection is higher in the uniform case than in the non-uniform case.

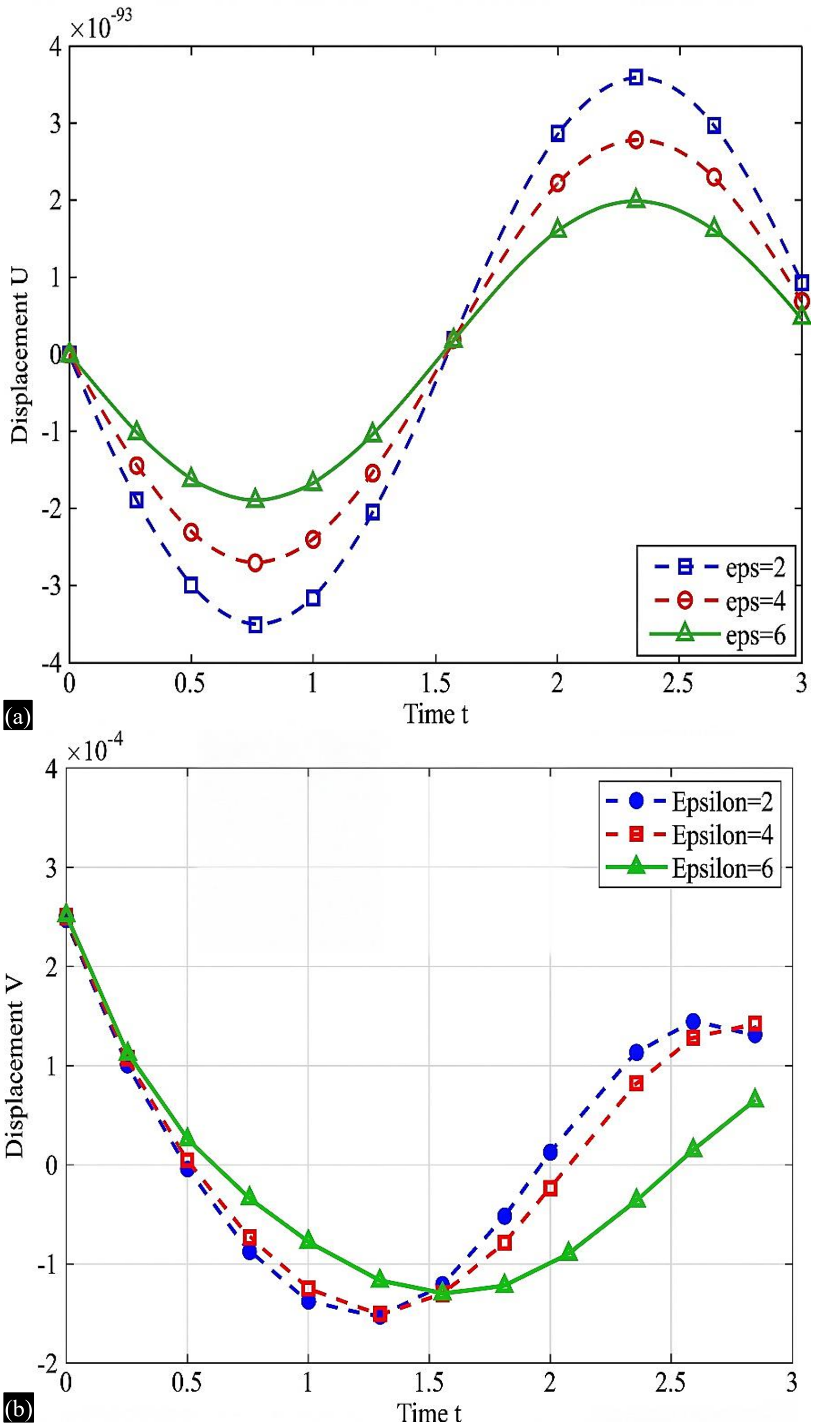

Figure 2. (a) Deflection profile of a uniform elastic beam resting on a Pasternak foundation under harmonic moving load for various values of damping coefficient (eps) with time-dependent boundary conditions; (b) Deflection profile of a non-uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of the damping coefficient (eps) with time-dependent boundary conditions.

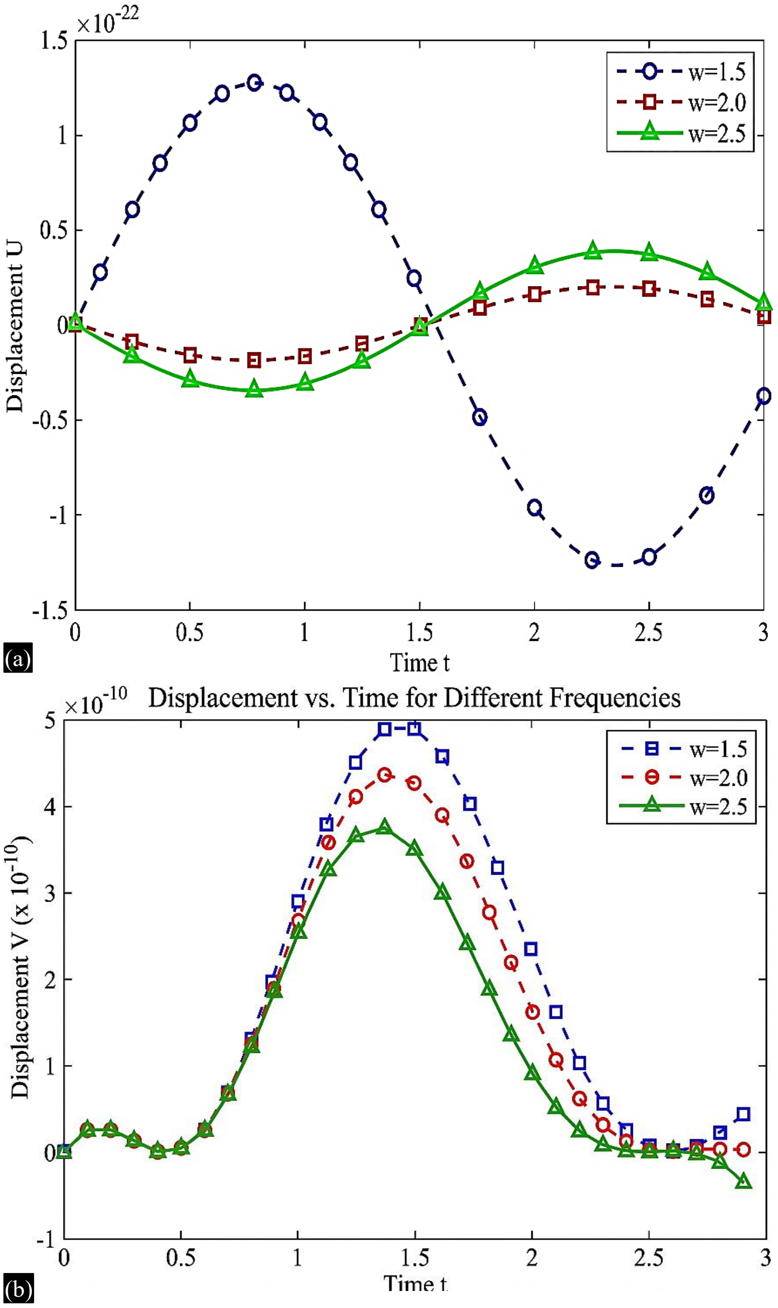

Figure 3. (a) Deflection profile of a uniform elastic beam resting on a Pasternak foundation under harmonic moving load for various values of load natural frequency (w) with time-dependent boundary conditions; (b) Deflection profile of a non-uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of the load natural frequency (w) with time-dependent boundary conditions.

The effects of the load natural frequency on both uniform and non-uniform beams are shown in Figure 3(a) and (b), respectively. The graphs revealed that higher values of the parameter reduce the vibration of the beams, but the vibration is much higher in the uniform beam than in the non-uniform beam.

The response amplitudes of uniform and non-uniform elastic beams considering the effect of axial force with other structural parameters held constant are shown in Figure 4(a) and (b), respectively. From the graphs, it can be seen that higher values of the axial force decrease the response amplitudes of the beams. The graphs also revealed that there is a significant decrease in the vibration of the uniform at the beginning, followed by a slight decrease, which also repeats itself towards the end of the beam. For the non-uniform beam, there is a slight decrease in the vibration at the beginning, followed by a significant decrease towards the end of the beam. The deflection is higher in the uniform beam than in the non-uniform beam.

Figure 5(a) and (b) shows the response amplitudes of the uniform and non-uniform beams, respectively. As seen from the graphs, the response amplitudes decrease as the shear modulus increases for both uniform and non-uniform beams. The impact of the shear modulus on the uniform beam is high at the beginning and end but has no impact at the end of the beam, whereas for the non-form beam, there is little effect at the beginning and middle, but a large effect at the end of the beam. The deflection is much lower in the non-uniform beam compared to that of the uniform beam.

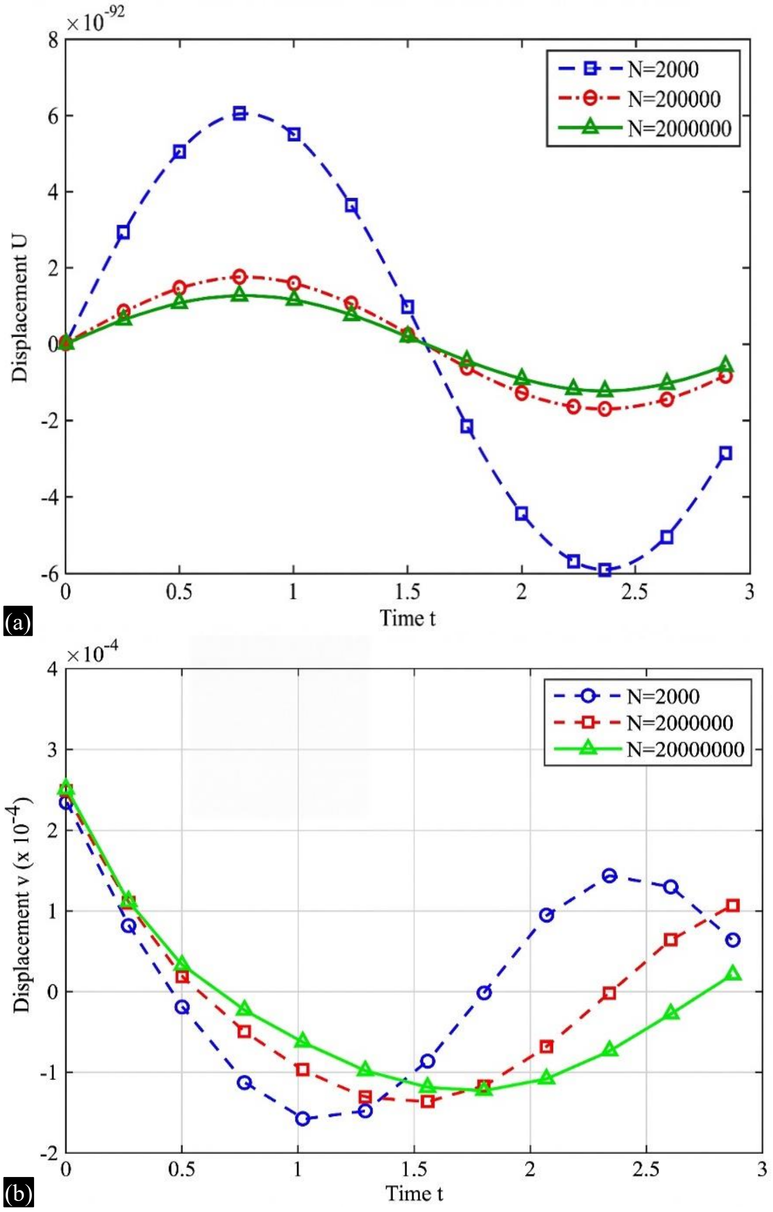

Figure 4. (a) Deflection profile of a uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of axial force (N) with time-dependent boundary conditions; (b) Deflection profile of a non-uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of axial force (N) with a time-dependent boundary conditions.

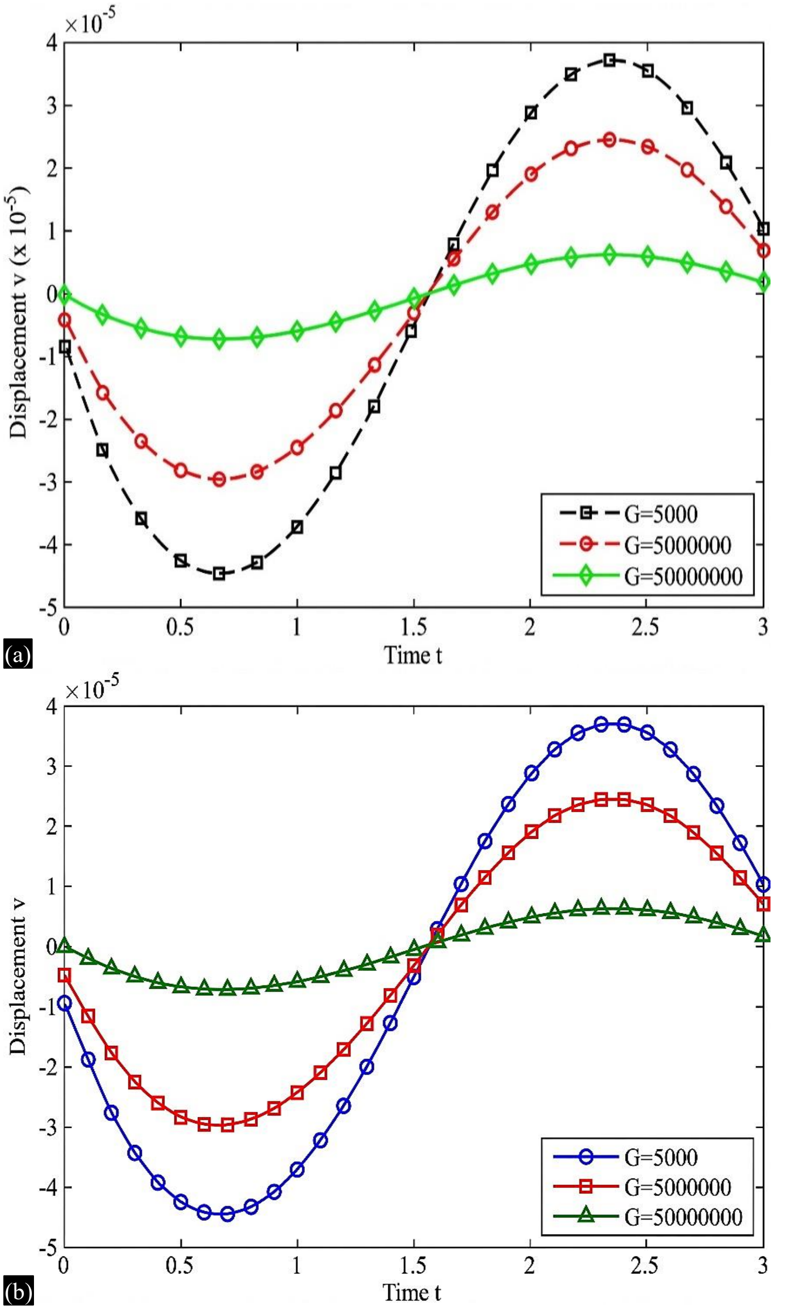

Figure 5. (a) Deflection profile of a uniform elastic beam resting on a Pasternak foundation under harmonic moving load for various values of shear modulus (G) with time-dependent boundary conditions; (b) Deflection profile of a non-uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of the shear modulus (G) with time-dependent boundary conditions.

The impact of the velocity of the moving load on both uniform and non-uniform beams is depicted in Figure 6(a) and (b), respectively. The results show that higher load velocities reduce the deflection profiles of the beams. The results also revealed that the parameter had a significant effect on the uniform beam compared to that on the non-uniform beam.

Finally, Figure 7 shows a comparison of the response amplitudes of the uniform and non-uniform beams. The results show that for fixed values of all the structural parameters, the deflection profile of the uniform beam is much higher than that of the non-uniform beam.

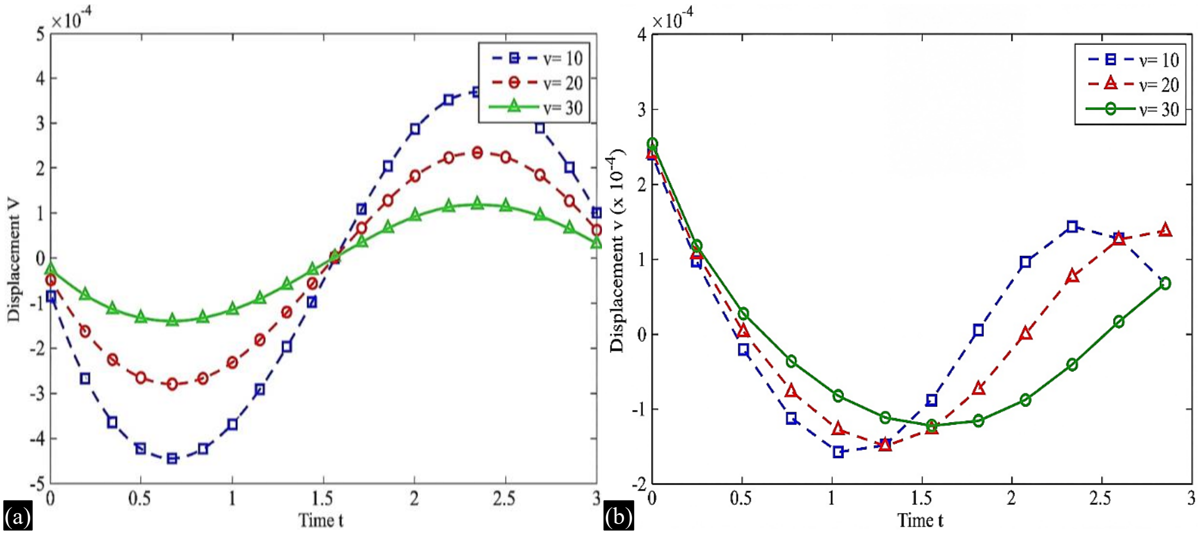

Figure 6. (a) Deflection profile of a uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of load velocity (v) with time-dependent boundary conditions; (b) Deflection profile of a non-uniform elastic beam resting on a Pasternak foundation under a harmonic moving load for various values of load velocity (v) with time-dependent boundary conditions.

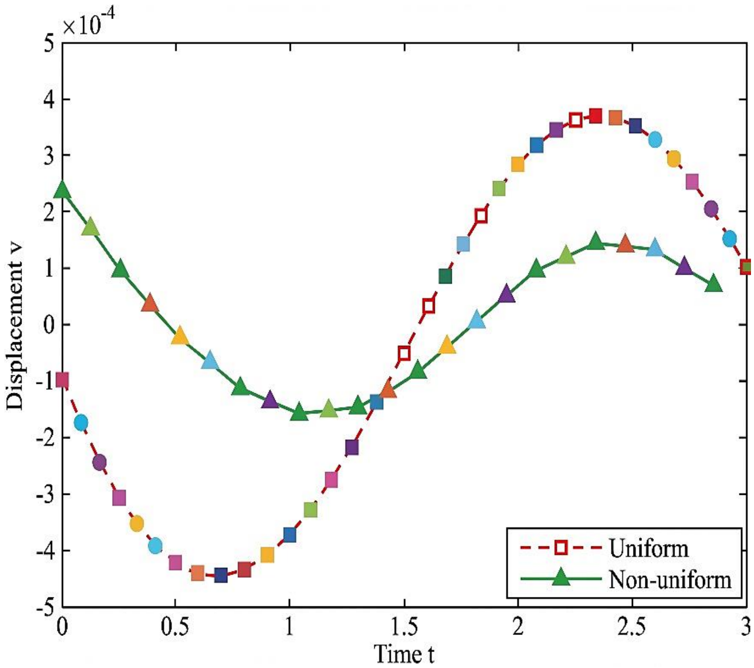

Figure 7. Comparison of deflection profiles of uniform and non-uniform elastic beams resting on a Pasternak foundation under a harmonic moving load with time-dependent boundary conditions.

CONCLUSION

In this study, the dynamic response to harmonically varying moving loads of both uniform and non-uniform elastic beams resting on a bi-parametric foundation with time-dependent boundary conditions was examined. The damping term is incorporated into the governing equations that describe the dynamical system. The moment of inertia and mass per unit length of the beam, which form the beam properties, were considered to be constant and also vary with position x along the span L of the beam. The solutions in plotted form were obtained using the Mindlin-Goodman method, Galerkin's method, Fourier sine transformation, integral transformation, and convolution integral. The findings of this study are highlighted below.

As the load natural frequency, load velocity, and damping coefficient increase, the displacement response of both uniform and non-uniform beams with time-dependent boundary conditions and under the action of a harmonic moving load decreases. The displacement responses are higher in the uniform beam than in the non-uniform beam for all the structural parameters considered. Increases in the values of the shear modulus, foundation modulus, and axial force lead to a decrease in the response amplitude of the beam with time-dependent boundary conditions and resting on a bi-parametric foundation. In addition, the displacement responses are higher in the uniform beam than in the non-uniform beam for all the structural parameters considered. The effects of the damping coefficient, load natural frequency, and load velocity were more noticeable on the dynamical system than those of other parameters. Lastly, the findings indicate that in all cases of the effects of the structural parameters on the dynamical system, the response amplitudes are higher in the uniform beam than in the non-uniform beam; thus, resonance is reached earlier in the uniform beam than in the non-uniform beam. In addition, for both uniform and non-uniform beams and all the structural parameters considered, the effects of the velocity of the moving load, load natural frequency, and damping coefficient were more noticeable than those of the other structural parameters. Hence, to reduce the tragedy of the resonance phenomenon and guarantee safety, designers should consider the cause of high vibrations during the design stage, as revealed by the findings.

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