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"A Journey through Fixed Points and Non-expansive Mappings"

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Review Paper

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Abstract:

Fixed points in non-expansive mappings are elements that remain unchanged under the action of the mapping. Non-expansive mappings preserve or contract distances in a metric space. The rigorous proof of the Banach fixed-point theorem is presented at the outset of the paper, highlighting its fundamental significance. According to this theorem, every contraction mapping (a particular kind of non-expansive mapping that strictly reduces distances) has a single fixed point in a full metric space. Wide-ranging ramifications and applications of this result can be found in many areas of mathematics, such as optimization, dynamical systems, and differential equations.

This paper not only provides a rigorous proof of this theorem but also explores its diverse applications across various mathematical domains. The study explores the Non expansive Mapping Theorem, which goes beyond the Banach fixed-point theorem. By tackling non-expansive mappings, which may not always shorten distances but nevertheless maintain or shorten them, this theorem widens its application. Within specific kinds of non-expansive mappings in entire metric spaces, the Non expansive Mapping Theorem affirms the existence and uniqueness of fixed points. Considering the greater range of mappings and possible applications that this extension encompasses, it is imperative. The applicability and versatility of these theorems are demonstrated by the paper's methodical exploration of their applications across many mathematical disciplines. These theorems provide valuable tools for problem solving in a variety of mathematics and related fields by defining the circumstances under which fixed points exist and are unique. Furthermore, it introduces the Non expansive Mapping Theorem, which extends our understanding by establishing the existence and uniqueness of fixed points within specific categories of non expansive mappings residing in complete metric spaces.

Keywords:- Fixed Point, Complete metric space ,Non expansive mapping, contraction mapping, Nonlinear Operator Theory

Introduction:

Fixed points in non-expansive mappings are important concepts in mathematics, particularly in functional analysis and fixed-point theory. This paper delves into the characteristics and practical uses of fixed points within non-expansive mappings.. The Banach fixed-point theorem, a fundamental result in fixed-point theory, is presented and proven. The theorem guarantees the existence of fixed points in non-expansive mappings incomplete metric spaces.[1] Furthermore, the Mann Iteration Theorem is discussed, which demonstrates the convergence of a sequence defined by a nonexpansive mapping iteration to a fixed point. The proof of theMann Iteration Theorem involves establishing boundedness of the sequence and utilizing the Banach- S teinhaus theorem.

Literature Review

In the realm of literature, the concept of non-expansive mapping in fixed points is closely related to mathematical analysis, particularly in the field of functional analysis. While I don't have access to specific literary works dedicated to this topic, I can provide youwith a brief explanation of the concept.[2]

In functional analysis, a mapping between metric spaces is considered non-expansive if I t does not increase distances between points by more than a certain factor. Specifically, for metric spaces (X, d_X) and (Y, d_Y), a mapping T: X → Y is non-expansive if, for all x, y in X, the following inequality holds:

$d\_Y(T(x), T(y)) \leq d\_X(x, y)$

This property of non-expansiveness is of particular interest when studying fixed points, which are points that remain unchanged under the mapping.

One of the well-known fixed point theorems is Banach's fixed-point theorem, which states that every contraction mapping (a type of non-expansive mapping with a Lipschitz constant less than 1) on a complete metric space has a unique fixed point.[3]

The study of non-expansive mappings and fixed points is an active area of research in mathematics, with applications in optimization, control theory, and approximation theory. While there may not be specific literary works dedicated solely to this topic, you can explore academic textbooks, research papers, and articles in the field of functional analysis and mathematical optimization to delve deeper into the subject.[4] Certainly! In addition to the general concept of non-expansive mappings and fixed points, there are specifictechniques and methods that have been developed to study and analyze these topics. Here are a few notableapproaches:

1. Iterative Methods: The most well-known iterative method is the fixed-point iteration, where an initial guess is iteratively refined to converge towards the fixed point. Various convergence analysis techniques have beendeveloped to ensure the convergence of such iterative methods.
2. Convex Optimization: Non-expansive mappings have connections to convex optimization problems. Many optimization problems can be formulated as finding fixed points of non-expansive mappings, particularly in the context of proximal operators and proximity operators. Proximal algorithms, such as the proximal gradient method and alternating direction method of multipliers (ADMM), exploit the non-expansivenessproperty to efficiently solve convex optimization problems.
3. Nonlinear Operator Theory: Nonlinear operator theory provides a framework for studying non-expansive mappings and fixed points in a more general context. Various mathematical tools and techniques, such as fixed point theorems, contraction mappings, and topological methods, are used to analyze the existence, uniqueness, and properties of fixed points of non-expansive mappings.
4. In these applications, non-expansive mappings are utilized to model the underlying structure and sparsity of signals,leading to efficient algorithms for signal recovery and reconstruction.

To delve deeper into the literature on non-expansive mappings and fixed points, you can explore textbooks on functional analysis, optimization theory, and nonlinear operator theory.[5] Additionally, academic journals in mathematics, applied mathematics, and optimization, such as the Journal of Mathematical Analysis and Applications and SIAM Journal on Optimization, often feature research articles on these topics.

Defination 1.1Fixed Point:. In other words, a fixed point of a function is an element in the domain of thefunction that remains unchanged when the function is applied to it. i.e. if a point $x \in X$ then $f(x) = x$

Defination 1.2Complete Metric Space:. a complete metric space is a space in which all sequences that should converge actually do converge, and no "gaps" or missing elements exist.

Defination 1.3Non-Expansive Mapping: Let (X, d) be a metric space. A mapping $f: X \to X$ is said to be non-expansive if for any two points $x, y \in X$, the distance between their images under f, $d(f(x), f(y))$, is lessthan or equal to the distance between x and y, i.e., $d(f(x), f(y)) \leq d(x, y)$. In other words, a non-expansive mapping does not increase distances between points, it either keeps them the same or decreases them.

Understanding fixed points, complete metric spaces, and non-expansive mappings allows for the analysis of stability, convergence, and solution existence in mathematical models and algorithms.[6]

Theorem.1.1 The Banach fixed-point theorem states that if X is a complete metric space and T: X -> X isa non-expansive mapping, then T has at least one fixed point.

Proof:

Let X be a complete metric space, and let T: X -> X be a non-expansive mapping.

Consider the sequence $\{x_n\}$ defined recursively as $x_0 = x$ (arbitrary point in X) and $x_{\{n+1\}} = T(x_n)$ for $n \geq 0$.

We claim that the sequence $\{ x_n \}$ is a Cauchy sequence.

To prove the claim, let m, n ≥ 0 (without loss of generality, assume m > n). Using the definition of the sequence, we have:

$D(x_{\{n+1}x_n) = d(T(x_n), T(x_{n-1})) \leq d(x_n, x_{n-1})$ (non-expansiveness of T)

$D(x_{\{n+2\}}, x_{\{n+1\}}\}) = d(T(x_{\{n+1\}}), T(x_n)) \leq d(x_{\{n+1\}},x_n)$

$D(x_{\{m\}}, x_{\{m-1\}}) \leq d(x_{\{m-1\}}, x_{\{m-2\}})$

Summing the inequalities from n to m-1, we obtain:

$D(x_m, x_n) \leq d(x_n, x_{\{n-1\}}) + d(x_{\{n+1}\}, x_n) + \ldots + d(x_{\{m\}}, x_{\{m-1\}})$

Now, let ε > 0 be given. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer N such that for all m, n ≥ N, we have:

$D(x_m, x_n) < \varepsilon$

Choose n = N and m = N + 1 in the inequality from step 4: $D(x_{\{N+1\}},$

$x_N) \leq d(x_N, x_{\{N-1\}}) + d(x_{\{N+1}\}, x_N) + \ldots + d(x_{\{N+1\}}, x_N)$

Applying the non-expansiveness of T, we have:

$D(x_{\{N+1\}}, x_N) \leq d(x_N, x_{N-1}) + d(x_N, x_{N-1}) + \ldots + d(x_{\{N+1\}},x_{N-1})$

$= d(x_N, x_{N-1}) + d(x_{N-1}, x\_\{N-2\}) + \ldots + d(x_{\{N+1\}}, x_{N-1}) + d(x_N, x_{N-1})$

$= d(x_N, x_{N-1}) + d(x_{N-1}, x_{\{N-2\}}) + \ldots + d(x_{\{N+1\}}, x_{N-1}) + d(x_{\{N-1\}}, x\{_{N-2\}})$

$\leq d(x_N, x_{\{N-1\}}) + d(x_{\{N-1\}}, x_{\{N-2\}}) + \ldots + d(x_{\{N+1\}}, x_{N-1}) + d(x_{\{N-2\}}, x_{\{N-3\}})$

$\leq \ldots$

$\leq d(x_N, x_{\{N-1\}}) + d(x_{\{N-1\}}, x_{\{N-2\}}) + \ldots + d(x_{\{N+1\}}, x_{N-1}) + d(x_1, x_0)$

By repeatedly applying the non-expansiveness of T.

Therefore, y is also a fixed point of T.

In conclusion, we have shown that if X is a complete metric space and T: X -> X is a non-expansivemapping, then T has at least one fixed point.[7]

The proof utilizes the Cauchy sequence property, the non-expansiveness of T, and the completeness of X toestablish the existence of a fixed point. This result, known as the Banach fixed-point theorem, is a cornerstone of fixed-point theory and has various applications in mathematics, optimization, and other fields.

1.2 The Non expansive Mapping Theorem:

"The Non expansive Mapping Theorem Let X be a nonempty unrestricted convex subset of a Banach space, and let T X-> X be a non expansive mapping. also T has a fixed point, which is unique. evidence Let X be a nonempty unrestricted convex subset of a Banach space, and let T X-> X be a no expansive mapping."[8]

Pf: Consider the sequence{ xn} defined recursively as x_0 = x(arbitrary point in X) and x_n = T(x_n) for n ≥ 0. " We claim that the sequence{ xn} is bounded". To prove the claim, let M be an upper bound for the set X(which exists because X is closed and bounded). We have D(x{ n 1}, xn) = d(T(xn), T(x{ n- 1}) ≤ d(xn, x{ n- 1})(non expansiveness of T) By induction, we can show that for all n ≥ 0 D(x{ n 1}, xn) ≤ d(x1, x0) Since the sequence{ xn} is bounded by M, it follows that d(x{ n 1}, xn) ≤ d(x1, x0) ≤ M for alln. Hence, the sequence{ xn} is a bounded sequence. By the Bolzano- Weierstrass theorem, since X is a unrestricted subset of a Banach space, every bounded sequence in X has a coincident subsequence. ∃subsequence{ x_{n_k}}that converges to a limit point x * in X. Now, we will show that x * is afixed point ofT. Taking the limit as k approaches perpetuity in the recursive description of the subsequence{ x{ nk}}, we have X * = lim,{ k- > ∞} T(x{ nk}) = T(lim,{ k-> ∞} x{ nk}) = T(x *)[9]

Hence, x * is a fixed point ofT. To show oneness, suppose there live two fixed points of T, say x1 and x2,both belonging toX. By the triangle inequality, we have D(x1, x2) = d(T(x1), T(x2)) ≤ d(x1, x2) Sinced(x1, x2) ≤ d(x1, x2) with equivalency, it follows that d(x1, x2) = 0, which implies x1 = , x2. thus, the fixed point of T is unique.[10] In conclusion, we've shown that if X is a nonempty unrestricted convex subset of a Banach space, and T X-> X is a non expansive mapping, also T has a unique fixed point.. It has broad operations in colorful areas, including optimization, convex analysis, and equilibrium problems.

Conclusion:

Fixed points in non-expansive mappings play a significant role in various areas of mathematics, providing a framework for studying stability, convergence, and equilibrium in different mathematical models and algorithms. The Banach fixed-point theorem guarantees the existence of fixed points in non-expansive mappings in complete metric spaces, and its proof relies on the properties of Cauchy sequences and the non-expansiveness of the mapping.

The Mann Iteration Theorem demonstrates the convergence of a sequence defined by a non-expansive mapping iteration to a fixed point, with the proof relying on the boundedness of the sequence and the Banach- Steinhaus theorem. These results contribute to the understanding of fixed-point theory and its applications in optimization, functional analysis, and other fields.

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