STM JOURNALS

Recent Trends in
Mathematics

Volume 1, Issue 1, 2024
January—June
DOI (Journal): 10.37591/RTM

https://journals.stmjournals.com/rtm

Research RTM

Advance Fixed Point Theorems and Its Application to Ordinary and Fractional Differential Equations

Shirish Prabhakarrao Kulkarni[1,*], Soni Pathak[2], Gooty Rohan[3]

Abstract

In the present research, advanced results on the fixed point theorem are applied to typical boundary value problems. Finding a differential equation's solution under certain boundary conditions is the goal of typical boundary value topics. The article appears to extend new fixed point results to a new context, likely involving fractional operators with unique kernels, notably the Caputo-type fractional operator. This extension involves applying the fixed point theorem to a broader set of problems. Caputo fractional derivatives are a generalization of ordinary derivatives to non-integer orders, commonly used in fractional calculus. This extends the scope to fractional boundary value problems, where the differential equation involves fractional derivatives, and the conditions are given in terms of fractional order. Integral type boundary conditions suggest that the conditions involve integrals, which could be part of the fractional differential equation or the boundary conditions. We have used inequalities on a triplet (U, d, T) and quatern (U, d, T, θ). We have developed a novel class of mappings and investigated a fixed point criterion for them, using Geraghty contraction as inspiration. In addition, we demonstrated two applications: one with singular kernels for fractional derivatives in a system of nonlinear differential equations and the other with a two-point boundary value problem of a second order ordinary differential equations.

Keywords: Fractional differential equations, Ordinary differential Equations, Generalized α-p-ϖ-contractions, Weakly contractive mapping, Geraghty function, θ-orbital permissible

INTRODUCTION

Fixed point theorem is a fundamental mathematical concept with applications in various fields, including functional analysis, optimization, and the study of iterative algorithms. It has been extensively employed in demonstrating the existence and uniqueness of solutions for various mathematical models, including integral equations and differential equations [1-3]. Researchers may have made advancements in extending and applying fixed point theorems to new mathematical frameworks, such as fractional calculus [4, 5]. Fractional calculus deals with derivatives and integrals of non-integer order, providing a more quiet and flexible approach to modeling complex systems. The mention of recent significant findings suggests that researchers have made breakthroughs in understanding and solving fractional differential and integral equations.

These findings could include new solution techniques, stability analyses, or applications to real-world problems. On the other hand, numerous publications have been reported since Banach's [6] remarkable fixed point result. We peak the significant papers by Geraghty [7], Boyd and Wong [8], Jaggi [9], Rhoades [10], and Dass and Gupta [11] among them.

***Author for Correspondence**
Shirish Prabhakarrao Kulkarni
E-mail: shirishpk@gmail.com

[1]Assistant Professor, Ajeenkya D.Y. Patil University (SOE), Charholi Budruk, Pune, Maharashtra, India
[2]Professor, Chhatrapati Shivaji Maharaj University, Panvel, Navi Mumbai, Maharashtra, India
[3]Assistant Professor, Ajeenkya D.Y. Patil University (SOE), Charholi Budruk, Pune, Maharashtra, India

Received Date: January 24, 2024
Accepted Date: March 01, 2024
Published Date: June 03, 2024

Citation: Shirish Prabhakarrao Kulkarni, Soni Pathak, Gooty Rohan. Advance Fixed Point Theorems and Its Application to Ordinary and Fractional Differential Equations. Recent Trends in Mathematics. 2024; 1(1): 1–15p.

For a non-empty set U operative through a metric d, we present a family of auxiliary functions

$p: U \times U \to [0, 1)$ such that

$$\lim_{n\to\infty} p(\mu_n, \lambda_n) = 1 \to \lim_{n\to\infty} d(\mu_n, \lambda_n) = 0$$

for all sequences $\{\mu_n\}$ and $\{\lambda_n\}$ in U that the sequence $\{d(\mu_n, \lambda_n)\}$ is decreasing and convergent. The function family, defined above, is represented by $A(U)$ (see, e.g., [12, 13]).

Example 1.1 Let $a_1, a_2 : \mathrm{R} \times \mathrm{R} \to [0, 1)$, defined by

$a_1(\mu, \lambda) = k$ for some $k \in (0, 1)$;

$a_2(\mu, \lambda) = \frac{t}{t+\mu^2+\lambda^2}$ for some $t \geq 0$

Then $a_1, a_2 \in A(\mathrm{R})$.

We reserve the letter G for all functions $\delta : [0, \infty) \to [0, 1)$ so that

$\delta(t_n) \to 1$ implies $t_n \to 0$.

A function $\delta \in G$ is also named Geraghty function.

Example 1.2 On a metric space (U, d), we define a function $a : U \times U \to [0, 1)$ by

$a(\mu, \lambda) = \delta\, d(\mu, \lambda)$,

where $\delta \in G$. For sequences $\{\mu_n\}$, $\{\lambda_n\}$ in U, if $\lim_{n\to\infty} p(\mu_n, \lambda_n) = 1$, then

$\lim_{n\to\infty} d(\mu_n, \lambda_n) = 1$ Thus

$\lim_{n\to\infty} d(\mu_n, \lambda_n) = 0$

This implies that $a \in A(U)$.

We introduce a family of continuous, non-decreasing auxiliary functions ϖ from $[0, \infty)$ to $[0, \infty)$ denoted by the letter Φ.

$\varpi(q) = 0$ if and only if $q = 0$.

From now on, a triplet (U, d, T) represents a structure:
1. A non-empty set U;
2. A metric on d so that (U, d) is complete;
3. T is a self-mapping on U.

We also define the following inequalities on a triplet (U, d, T):

(D_1) $\varpi(d(T\mu, T\lambda)) \leq \varpi(d(\mu, \lambda)) - \psi(d(\mu, \lambda))$ for each $\mu, \lambda \in U$, where $\varpi, \psi \in \Phi$;

(D_2) $d(T^{\mu}, T\lambda) \leq \delta(d(\mu, \lambda)) d(^{\mu}, \lambda)$ for each $\mu, \lambda \in U$, where $\delta \in G$.

Theorem 1.3 (Dutta and Choudhury [14]) *On* (U, d, T), *if* (D_1) *is fulfilled, then T has a fixed point.*

Theorem 1.4 (Geraghty [7]) *On* (U, d, T), *if* (D_2) *is fulfilled, then T has a fixed point*. Popescu [15] recommended a modification on (triangular-) α-a permissible mappings, defined in [16, 17], as follows.

Definition 1.5 ([15]) On a triplet (U, d, T), for a mapping $\theta: U \times U \to [0, \infty)$, T is called θ -orbital permissible if

$\theta(q, Tq) \geq 1 \Rightarrow \theta(Tq, T^2q) \geq 1.$

In addition, if the inequality

$\theta(p, q) \geq 1$ and $\theta(q, Tq) \geq 1 \Rightarrow \theta(p, Tq) \geq 1$

is fulfilled, then the θ -permissible mapping T is named triangular θ -orbital permissible.

Observe that every mapping that is θ -permissible is also θ -orbital permissible. See e.g. for additional information and counterexamples [15–19].

If a mapping exists, in addition to the triplet (U, d, T) structure $\theta: U \times U \to [0, \infty)$, we epitomize it by a quatern (U, d, T, θ). It is vibrant that (U, d, T, θ) decreases to (U, d, T) in event of $\theta(p, q) = 1$ for all $p, q \in U$.

Definition 1.6 ([19]) On a edifice (U, d, T, θ), we approximately that U is θ -regular if the succeeding condition is satisfied: (θ -Regular) If $\{\mu_n\}$ is a sequence in U such that $\theta(\mu_n, \mu_{n+1}) \geq 1$ for all $n \in$ N and $\mu_n \to \mu \in U$ as $n \to \infty$, then there exists a subsequence $\{\mu_{nk}\}$ of $\{\mu_n\}$ such that $\theta(\mu_{nk}, \mu) \geq 1$ for all k.

Meanwhile, due to the outcomes of using fractional operators in modelling, one of the rapidly expanding fields of study is the fractional calculus, which expands the integer order integration and differentiation to any order [20–23].

Recent discoveries of fractional operators have been proposed by some researchers as a way to better comprehend some real-world challenges. We highlight the operators that are thought to be among these in [24–28].

On the other hand, several researchers used fixed point theorems to examine the existence and uniqueness of solutions to differential/integral equations using fractional operators. With respect to this issue, we refer to [28–37].

The purpose of this study is to prove a few new fixed point theorems and demonstrate the existence and uniqueness of solutions to a number of differential equations of both fractional and integer order.

MAIN RESULTS

This segment will designate a new class of mappings and appearance into fixed point criterion for them, drawing motivation from Geraghty contraction.

Definition 2.1 On a quatern (U, d, T, θ), we define the following inequalities:

(D_3) $\theta(\mu, \lambda)\, \varpi\, (d(T\mu, T\lambda)) \leq p(\mu, \lambda)\, \varpi\, (R(\mu, \lambda))$ for all $\mu, \lambda \in U$,

(D_4) $\theta(\mu, \lambda)\, \varpi\, (d(T\mu, T\lambda)) \leq p(\mu, \lambda)\, \varpi\, (J(\mu, \lambda))$ for all $\mu, \lambda \in U$,

(D_5) $\theta(\mu, \lambda)\, \varpi\, (d(T\mu, T\lambda)) \leq p(\mu, \lambda)\, \varpi\, (M(\mu, \lambda))$ for all $\mu, \lambda \in U$,

where $p \in A(U)$ and $\varpi \in \Phi$, and

$$R(\mu, \lambda) = (max\{\frac{d(\mu, T\mu)d(\lambda, T\lambda)}{d(\mu, \lambda)}, d(\mu, \lambda), d(\mu, T\mu), d(\lambda, T\lambda), \frac{d(\mu, T\lambda) + d(\lambda, T\mu)}{2}\} \tag{2.1}$$

$$J(\mu, \lambda) = (max\{\frac{d(\mu, T\mu)d(\lambda, T\lambda)}{d(\mu,\lambda)}, d(\mu, \lambda)\} \quad (2.2)$$

$$M(\mu, \lambda) = (max\{ d(\mu, \lambda), d(\mu, T\mu), d(\lambda, T\lambda)\} \quad (2.3)$$

We say that T is Jaggi type θ -p-ϖ -contraction (respectively, generalized Jaggi type θ -p-ϖ -reduction) if (*D*3) (correspondingly, (*D* 4)) is satisfied. A charting T will be called θ -p-ϖ - reduction if (*D* 5) is fulfilled.

Theorem 2.2 On a quatern (U, d, T, θ), if the following assumptions hold:

1. discrimination (D3) holds;
2. T is unceasing and systems a trilateral θ-orbital permissible;
3. around happens $\mu_0 \in U$ such that $\theta(\mu_0, T\mu_0) \geq 1$;

Then T has a fixed point.

Proof: On account of (iii), there is $\mu_0 \in U$ with

$\theta(\mu_0, T\mu_0) \geq 1$.

Define an iterative sequence $\{\mu_n\}$ by $\mu_n = T\mu_{n-1}$ for all $n \in N$. Assume that for some positive integer *k*, we have $\mu_k = \mu_{k+1}$. This implies that $T\mu_k = \mu_{k+1} = \mu_k$, that is μ_k is a fixed point of *T*. Thus, we shall assume that $\mu_n \neq \mu_{n+1}$ for all *n* = 0, 1, 2, .

On account of inequality (D_3), for all $n \in N$, we have

$$\varpi(d(\mu_n, \mu_{n+1})) \leq \theta(\mu_{n-1}, \mu_n)\, \varpi(d(\mu_n, \mu_{n+1}))$$
$$= \theta(\mu_{n-1}, \mu_n)\, \varpi(d(T\mu_{n-1}, T\mu_n))$$
$$\leq p(\mu_{n-1}, \mu_n)\, \varpi(R(\mu_{n-1}, \mu_n))$$
$$< \varpi(R(\mu_{n-1}, \mu_n)). \quad (2.4)$$

On the other hand

$$R(\mu_{n-1}, \mu_n) = max\{\frac{d(\mu_{n-1}, T\mu_{n-1})d(\mu_n, T\mu_n)}{d(\mu_{n-1}, \mu_n)}, d(\mu_{n-1}, \mu_n), d(\mu_{n-1}, T\mu_{n-1}), d(\mu_n, T\mu_n), \frac{d(\mu_{n-1}, T\mu_n)+d(\mu_n, T\mu_n)}{2}$$

$$R(\mu_{n-1}, \mu_n) = max\{ d(\mu_n, \mu_{n+1}), d(\mu_{n-1}, \mu_n), d(\mu_n, T\mu_{n+1}), \frac{d(\mu_{n-1}, \mu_{n+1})+d(\mu_n, \mu_n)}{2}\}$$

$$R(\mu_{n-1}, \mu_n) = max\{ d(\mu_n, \mu_{n+1}), d(\mu_{n-1}, \mu_n), \frac{d(\mu_{n-1}, \mu_{n+1})}{2}\}$$

$$R(\mu_{n-1}, \mu_n) \leq max\{ d(\mu_n, \mu_{n+1}), d(\mu_{n-1}, \mu_n), \frac{d(\mu_{n-1}, \mu_n)+d(\mu_n, \mu_{n+1})}{2}\}$$

$$R(\mu_{n-1}, \mu_n) = max\{ d(\mu_n, \mu_{n+1}), d(\mu_{n-1}, \mu_n)\}$$

If $R(\mu_{n-1}, \mu_n) = \{d(\mu_n, \mu_{n+1})\}$

Employing (2.4), we conclude that

$$\varpi(d(\mu_n, \mu_{n+1})) < \varpi(R(\mu_n, \mu_{n+1}))$$
$$\varpi(d(\mu_n, \mu_{n+1})) = \varpi(d(\mu_n, \mu_{n+1}))$$

A contradiction. So, we accomplish that, for all $n \in$ N,

$$R(\mu_{n-1}, \mu_n) = d(\mu_{n-1}, \mu_n) \tag{2.5}$$

Now, from (2.4) and (2.5), we get that

$$\varpi(d(\mu_n, \mu_{n+1})) < \varpi(d(\mu_{n-1}, \mu_n))$$

Using the monotony of ϖ implies that, for all $n \in$ N,

$$d(\mu_n, \mu_{n+1})) < d(\mu_{n-1}, \mu_n)$$

So, the sequence $\{d(\mu_n, \mu_{n+1})\}$ is non-negative and decreasing. Eventually, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(\mu_n, \mu_{n+1}) = r$$

Thereafter, we illustrate that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, from (2.4) and (2.5), we have

$$0 < \frac{\varpi(d(\mu_n, \mu_{n+1}))}{\varpi(d(\mu_{n-1}, \mu_n))} \leq p(\mu_{n-1}, \mu_n)$$

which suggests that

$$\lim_{n\to\infty} d(\mu_{n-1}, \mu_n) = 0$$

It yields $r = 0$ that is a contradiction. Hereby,

$$\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = 0$$

The sequence $\{\mu_n\}$ is important (Cauchy). Receive, on the opposing, that the iterative sequence $\{\mu_n\}$ is not important. Then here happens $\in > 0$ such that, for all $k \in$ N, we can find $m_k \geq n_k > k$ such that

$$d(\mu_{nk}, \mu_{mk}) \geq \in$$

In adding, it may be expected that

$$d(\mu_{nk}, \mu_{mk-1}) < \in$$

By selecting m_k as small as possible. Hence, for each $k \in$ N, we find

$$\in \leq d(\mu_{nk}, \mu_{mk}) \leq d(\mu_{nk}, \mu_{mk-1}) + d(\mu_{mk-1}, \mu_{mk})$$
$$\in \leq \in + d(\mu_{mk-1}, \mu_{mk})$$

Letting $k \to \infty$, we get

$$d(\mu_{nk}, \mu_{mk}) = \in$$

Note that, for any $k \in$ N,

$$\varpi d(\mu_{nk+1}, \mu_{mk+1}) \leq \theta(\mu_{nk}, \mu_{mk}) \varpi\big(d(\mu_{nk+1}, \mu_{mk+1})\big)$$
$$\varpi d(\mu_{nk+1}, \mu_{mk+1}) \leq \theta(\mu_{nk}, \mu_{mk}) \varpi(d(T\mu_{nk}, T\mu_{mk}))$$
$$\varpi d(\mu_{nk+1}, \mu_{mk+1}) \leq p(\mu_{nk}, \mu_{mk}) \varpi(R(\mu_{nk}, \mu_{mk})) \tag{2.6}$$

Likewise, for any $k \in$ N, we have

$$R(\mu_{nk},\mu_{mk}) = max\{\frac{d(\mu_{nk},T\mu_{nk})d(\mu_{mk},T\mu_{mk})}{d(\mu_{nk},\mu_{nk})}, d(\mu_{nk},\mu_{mk})d(\mu_{nk},T\mu_{nk})d(\mu_{mk},T\mu_{mk}), \frac{d(\mu_{nk},T\mu_{mk})+d(\mu_{mk},T\mu_{nk})}{2}\}$$

$$R(\mu_{nk},\mu_{mk}) = max\{\frac{d(\mu_{nk},\mu_{nk+1})d(\mu_{mk},\mu_{mk+1})}{d(\mu_{nk},\mu_{mk})}, d(\mu_k,\mu_{mk}), d(\mu_{nk},\mu_{nk+1}), d(\mu_{mk},\mu_{mk+1}), \frac{d(\mu_{nk},\mu_{mk})+d(\mu_{mk},\mu_{nk+1})}{2}\}$$

$$R(\mu_{nk},\mu_{mk}) \leq max\{\frac{d(\mu_{nk},\mu_{nk+1})d(\mu_{mk},\mu_{mk+1})}{d(\mu_{nk},\mu_{mk})}, d(\mu_{nk},\mu_{mk}), d(\mu_{nk},\mu_{nk+1}), d(\mu_{mk},\mu_{mk+1}), \frac{d(\mu_{nk},\mu_{mk})+d(\mu_{mk},\mu_{nk+1})+d(\mu_{mk},\mu_{nk})+d(\mu_{nk},\mu_{nk+1})}{2}\}$$

Care in attention, the above inequality produces that

$$\lim_{k\to\infty} d(\mu_{nk},\mu_{mk}) = 0$$

$$\lim_{k\to\infty} R(\mu_{nk},\mu_{mk}) = \lim_{k\to\infty} d(\mu_{nk},\mu_{mk}) \tag{2.7}$$

On account of the triangular inequality and letting $k \to \infty$, we derive

$$\lim_{k\to\infty} d(\mu_{nk},\mu_{mk}) \leq \lim_{k\to\infty}(d(\mu_{nk},\mu_{nk+1}) + (\lim_{k\to\infty} d(\mu_{nk},\mu_{mk+1}) + (d(\mu_{mk+1},\mu_{mk})$$

$$\lim_{k\to\infty} d(\mu_{nk},\mu_{mk}) = \lim_{k\to\infty} d(\mu_{nk+1},\mu_{mk+1}) \tag{2.8}$$

Keeping the continuity of ϖ in mind and joining (2.6), (2.7), and (2.8), we get

$$\lim_{k\to\infty} \varpi(d(\mu_{nk},\mu_{mk})) \leq \lim_{k\to\infty} p(\mu_{nk},\mu_{mk}) \lim_{k\to\infty} \varpi d(\mu_{nk},\mu_{mk})$$

Since

$$\lim_{k\to\infty} d(\mu_{nk},\mu_{mk}) = \epsilon > 0$$

$$\lim_{k\to\infty} p(\mu_{nk},\mu_{mk}) = 1$$

Since p ∈ A(U) then

$$\lim_{k\to\infty} d(\mu_{nk},\mu_{mk}) = 0$$

It is a contradiction. As a outcome, the iterative sequence $\{\mu_n\}$ is Cauchy. Consequently, there exists $\mu^* \in X$ such that $\lim_{k\to\infty} \mu_n = \mu *$Regarding the continuity of T, we find

$$\lim_{k\to\infty} \mu_{n+1} = \lim_{k\to\infty} T\mu_n = T\mu *$$

Thus

$$T\mu = \mu *$$

Definition 2.3

On a structure (U, d, T, ϖ), we consider the following inequality:

(D_6) $\theta(\mu, \lambda) \varpi(d(T\varpi, T\lambda)) \leq p(\mu, \zeta) \varpi(Q(\mu, \lambda))$ for all $\mu, \lambda \in X$, where $p \in A(U)$, $\varpi \in \Phi$, and

$$Q(\mu,\lambda) = max\{\frac{d(\mu,T\mu)[1+d(\eta,T\eta)]}{1+d(\mu,\lambda)}, \frac{d(\lambda,T\lambda)[1+d(\mu,T\mu)]}{1+d(\mu,\lambda)}\} \tag{2.9}$$

A self-mapping T is named a comprehensive Dass–Gupta type θ -p-ϖ -reduction if (D_4) is satisfied

Theorem 2.4 On (U, d, T, θ), we assume that

i. inequality (D_6) holds;
ii. T is continuous and triangular θ-orbital permissible;
iii. there exists $\mu_0 \in$ U such that θ (μ_0, T μ_0) ≥ 1.

Then T has a fixed point.

Proof: From condition (iii), there exists $\mu_0 \in U$ such that

$$\theta(\mu_0, T\mu_0) \geq 1$$

We know that the sequence $\{\mu_n\}$ by $\mu_n = T\,\mu_{n-1}$ for all $n \in$ N. Supposing that, for nearly positive integer k, we take $\mu_k = \mu_{k+1}$. This suggests that $T\,\mu_k = \mu_{k+1} = \mu_k$, that is, μ_k is a fixed point of T. So, we can accept that $\mu_n \neq \mu_{n+1}$, n = 0, 1, 2....

Taking inequality (D_6) into account, we find

$$\varpi(d(\mu_n,\mu_{n+1})) \leq \theta(\mu_{n-1},\mu_n)\varpi(d(\mu_n,\mu_{n+1}))$$

$$\varpi(d(\mu_n,\mu_{n+1})) \leq \theta(\mu_{n-1},\mu_n)\varpi(d(T\mu_{n-1},T\mu_n))$$

$$\varpi(d(\mu_n,\mu_{n+1})) \leq p(\mu_{n-1},\mu_n)\varpi(d(\mu_{n-1},T\mu_n))$$

$$\varpi(d(\mu_n,\mu_{n+1})) \leq \varpi(Q(\mu_{n-1},\mu_n) \tag{2.10}$$

On the other hand,

$$Q(\mu_{n-1},\mu_n) = max\{\frac{d(\mu_{n-1},T\mu_{n-1})[1+d(\mu_n,T\mu_n)]}{1+d(\mu_{n-1},\mu_n)}, \frac{d(\mu_n,T\mu_n)[1+d(\mu_{n-1},T\mu_{n-1})]}{1+d(\mu_{n-1},\mu_n)}, d(\mu_{n-1},\mu_n)$$

$$Q(\mu_{n-1},\mu_n) = max\{\frac{d(\mu_{n-1},\mu_n)[1+d(\mu_n,\mu_{n+1})]}{1+d(\mu_{n-1},\mu_n)}, \frac{d(\mu_n,\mu_{n+1})[1+d(\mu_{n-1},\mu_n)]}{1+d(\mu_{n-1},\mu_n)}, d(\mu_{n-1},\mu_n)\}$$

$$Q(\mu_{n-1},\mu_n) = max\{\frac{d(\mu_{n-1},\mu_n)[1+d(\mu_n,\mu_{n+1})]}{1+d(\mu_{n-1},\mu_n)}, d(\mu_n,\mu_{n+1}), d(\mu_{n-1},\mu_n)\}$$

If $d(\mu_{n-1},\mu_n) \leq d(\mu_n,\mu_{n+1})$ then $Q(\mu_{n-1},\mu_n) = d(\mu_n,\mu_{n+1})$ applying (2,4), we deduce that

$$\varpi(d(\mu_n,\mu_{n+1})) < \varpi(Q(\mu_{n-1},\mu_n))$$

$$\varpi(d(\mu_n,\mu_{n+1})) = \varpi\big(d(\mu_n,\mu_{n+1})\big)$$

Which yields a contraction. Accordingly, we find

$$Q(\mu_{n-1},\mu_n) = d(\mu_{n-1},\mu_n) \tag{2,11}$$

Entirely n $\in$ N. On explanation of (2.4) and (2,5), we originate that

$$\varpi(d(\mu_n,\mu_{n+1})) < \varpi(d(\mu_{n-1},\mu_n))$$

Employing the monotonicity of ϖ ,we get

$$d(\mu_n,\mu_{n+1})) < d(\mu_{n-1},\mu_n)$$

for all $n \in$ N. we accept that $\{d(\mu_n, \mu_{n+1})\}$ is a non-negative and declining sequence. As an instant consequence, we realize that here happens $r \geq 0$ such that

$\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = r$. In pardon tracks, we declare that the limit $r = 0$. Supposing, on the conflicting, that $r > 0$. Then, from (2.4) and (2.5), we have

$$0 < \frac{\varpi(d(\mu_n,\mu_{n+1}))}{\varpi(d(\mu_{n-1},\mu_n))} \leq p(\mu_{n-1},\mu_n)$$

which implies that $\lim_{n\to\infty} p(\mu_n, \mu_{n+1}) = 1$. Since $p \in A(U)$,

$$\lim_{n\to\infty} d(\mu_{n-1}, \mu_n) = 0=r$$

It contradicts our assumption. Therefore

$$r=0=\lim_{n\to\infty} d(\mu_n, \mu_{n+1})$$

We will confirm that the sequence $\{\mu n\}$ is a fundamental (Cauchy) sequence in the result. Assume that $\{\mu n\}$ is not a fundamental sequence in the contradictory case. Henceforth, there exists $\in > 0$ such that, for all $k \in$ N, we can find $m_k \geq n_k > k$ such that

$$d(\mu_{n_k},\mu_{mk}) \geq \in$$

Also, choosing m_k as minor as possible, it may be supposed that

$$d(\mu_{n_k},\mu_{mk-1}) < \in$$

Therefore, for each $k \in$ N, we have

$$\in \leq d(\mu_{n_k},\mu_{mk}) \leq d(\mu_{n_k},\mu_{mk-1}) + d(\mu_{m_{k-1}},\mu_{mk})$$

$$\in \leq \in + d(\mu_{m_{k-1}},\mu_{mk})$$

As $k \to \infty$ in the inequality above, we get

$$\lim_{n\to\infty} d(\mu_{n_k},\mu_{mk}) = \in$$

Note that, for any $k \in N$

$$\varpi((d(\mu_{nk+1},\mu_{mk+1})) \leq \theta(\mu_{nk},\mu_{mk})\varpi(d(\mu_{nk+1},\mu_{mk+1}))$$

$$\varpi((d(\mu_{nk+1},\mu_{mk+1})) \leq \theta(\mu_{nk},\mu_{mk})\varpi(d(T\mu_{nk},T\mu_{mk}))$$

$$\varpi((d(\mu_{nk+1},\mu_{mk+1})) \leq p(\mu_{nk},\mu_{mk})\varpi((Q(\mu_{nk},\mu_{mk})) \tag{2,12}$$

Similarly, for any $k \in N$ we have

$$Q(\mu_{nk},\mu_{mk}) = \max\{\frac{d(\mu_{nk},T\mu_{mk})[1+d(\mu_{mk},T\mu_{mk})]}{1+d(\mu_{nk},\mu_{mk})}, \frac{d(\mu_{mk},T\mu_{mk})[1+d(\mu_{nk},T\mu_{nk})]}{1+d(\mu_{nk},\mu_{mk})}, d(\mu_{nk},\mu_{mk})$$

$$Q(\mu_{nk},\mu_{mk}) = \max\{\frac{d(\mu_{nk},\mu_{mk+1})[1+d(\mu_{mk},\mu_{mk+1})]}{1+d(\mu_{nk},\mu_{mk})}, \frac{d(\mu_{mk},\mu_{mk+1})[1+d(\mu_{nk},\mu_{nk+1})]}{1+d(\mu_{nk},\mu_{mk})}, d(\mu_{nk},\mu_{mk})$$

On the additional pointer, on explanation of the comment $\lim_{x\to\infty} d(\mu_{n_k}, \mu_{nk+1}) = 0$, we find

$$\lim_{k\to\infty} Q(\mu_{n_k}, \mu_{mk}) = \lim_{k\to\infty} d(\mu_{n_k}, \mu_{mk}) \tag{2,13}$$

On account of the triangular inequality and taking the limit as $n \to \infty$, we derive

$$\lim_{n\to\infty} d(\mu_{n_k}, \mu_{mk}) \le \lim_{k\to\infty} (d(\mu_{n_k}, \mu_{mk+1}) + d(\mu_{n_{k+1}}, \mu_{mk+1}) + d(\mu_{m_{k+1}}, \mu_{mk})$$

$$\lim_{n\to\infty} d(\mu_{n_k}, \mu_{mk}) = \lim_{k\to\infty} (\mu_{n_{k+1}}, \mu_{mk+1}) \tag{2,14}$$

Keeping the assumption $\lim_{k\to\infty} d(\mu_{n_k}, \mu_{mk}) = \in > 0$in mind, we deduce that

$$\lim_{k\to\infty} p(\mu_{n_k}, \mu_{mk}) = 1$$

Regarding that p ∈A(U), we get

$$\lim_{n\to\infty} d(\mu_{n_k}, \mu_{mk}) = 0$$

A contraction that, as a result the sequence $\{\mu_n\}$ is fundamental. Furthermore, there exist $\mu * \in U$ such that $\lim_{n\to\infty} \mu_n = \mu *$. since T is a continuous function, Therefore

$$\lim_{n\to\infty} \mu_{n+1} = \lim_{n\to\infty} T\mu_n = T\mu *$$

Thus $T\mu_n = \mu *$

Definition 2.5 On a quatern (U, d, T, θ), we define

$(D_3)^* \, \theta\,(\mu, \lambda)\, \varpi\, (d(T\mu, T\lambda)) \le p(\mu, \lambda)\, \varpi\, (R(\mu, \lambda))$ for all $\mu, \lambda \in U$,

where $p \in A(U)$ and $\varpi \in \Phi$, and $R(\mu, \lambda)$is defined as in (2.1), under the condition

$$\lim_{k\to\infty} p(\mu_n, \lambda_n) = 1 \Rightarrow \lim_{n\to\infty} d(T\mu_n, T\lambda_n) = 0$$

for all sequences $\{\mu_n\}$, $\{\lambda_n\} \subseteq U$ that $\theta\,(\mu_n, \lambda_n) \neq 0$ for all $n \in$ N. We say that *T is* a comprehensive Jaggi type θ -*p*- ϖ -contraction if $(D_3)^*$ is satisfied.

In its residence of the continuousness state, in Theorem 2.2, we suggest that *U* is θ -regular, as follows.

Theorem 2.6 On a quatern (U, d, T, θ), if the following assumptions hold:

i. inequality $(D_3)^*$ holds;
ii. X is θ -regular and T is continuous and forms triangular θ-orbital permissible;
iii. there exists $\kappa_0 \in X$ such that $\theta\,(\mu_0, T\mu_0) \ge 1$;

then T has a fixed point.

Proof: From condition (iii), there exists $\mu_0 \in X$ such that

$\theta\,(\mu_0, T\,\mu_0) \ge 1.$

Define the sequence $\{\mu_n\}$ by $\mu_n = T\,\mu_{n-1}$ for all $n \in$ N. Continuing along the relevant lines in the Theorem's proof 2.2, we know that the sequence $\{\mu_n\}$ is convergent to some $\mu^* \in X$ and $\theta\,(\mu_n, \mu_{n+1}) \ge 1$ for all $n \in$ N. Since the sequence *X* is θ -regular, there exists a subsequence

$\{\mu_{nk}\}$ with $\theta\,(\mu_{nk}, \mu^*) \ge 1$ for each $k \in$ N., we accept that

$$\theta(\mu_n, \mu) \ge 1 \text{ for all } n \in \mathrm{N} \tag{2.15}$$

Put on (2.15) for all $n \in$ N, we get

$$\varpi((d(\mu_{n+1}, T\mu *)) = \varpi(d(T\mu_n, T\mu *))$$

$$\varpi((d(\mu_{n+1}, T\mu *)) \leq \theta(\mu_n, \mu *)\varpi((d(T\mu_n, T\mu *))$$

$$\varpi((d(\mu_{n+1}, T\mu *)) \leq p(\mu_n, \mu *)\varpi((R(\mu_n, \mu *)) \quad (2.16)$$

Also, we have

$$R(\mu_n, \mu *) = max(\frac{d(\mu_n,T\mu_n)(\mu*,T\mu*)}{d(\mu_n,\mu*)}, d(\mu_n, \mu *), d(\mu_n, T\mu_n), d(\mu *, T\mu *), \frac{d(\mu_n,T\mu*)+d(\mu*,T\mu_n)}{2}\}$$

$$R(\mu_n, \mu *) = max(\frac{d(\mu_n,\mu_{n+1})(\mu*,T\mu*)}{d(\mu_n,\mu*)}, d(\mu_n, \mu *), d(\mu_n, \mu_{n+1}), d(\mu *, T\mu *), \frac{d(\mu_n,T\mu*)+d(\mu*,\mu_{n+1})}{d(\mu_n,\mu*)}\}$$

Since $\lim_{n\to\infty} d(\mu_n, \mu *) = 0$ then $\lim_{n\to\infty} R(\mu_n, \mu *) = d(\mu_n, T\mu *)$.Applying (2,10) and the continuity of ϖ ,we get

$\lim_{n\to\infty} p(\mu_n, \mu *) = 1$ and so

$d(\mu_n, T\mu *) = \lim_{n\to\infty} p(T\mu_n, T\mu *) = 0$

Therefore $T\mu *= \mu *$

APPLICATION

Applying Singular Kernels for Fractional Derivatives in a System of Nonlinear Differential Equations

Now we prove application let $\alpha \in (1,2]$.

Let α be a positive real number and Γ be a gamma function. For a continuous function f: [0, ∞) → R, the Caputo derivative of fractional order α is defined as

$$({}_0^cD^\alpha f)(s) = \frac{1}{\Gamma(n-)}\int_0^s (s-t)^{n-\alpha-1} f^{(n)}(t)\, dt,\ n = [\alpha] + 1.$$

Consider the following nonlinear fractional differential equation:

$$({}_0^cD^\alpha u)(s) = g(s, u(s)),\ s \in I \text{ and } 1 < \alpha \leq 2,$$

via the integral boundary condition

$$u(0) = 0,\ u(1) = \int_0^r u(t)dt,\ r \in (0,1),$$

where u ϵ C [0,1] and g : I x R→ R is a continuous function. We define the operator equation

T: C [0, 1]- C [0, 1] as follows:

$$T(u)(s) = \frac{1}{\Gamma()}\int_0^t (s-t)\,\alpha - 1\ g(t, u(t))\,dt - \frac{2s}{(2-r^2)} \cdot \frac{1}{\Gamma()}\int_0^1 (1-t)\,\alpha - 1 g(t, u(t))\,dt$$

$$+ \frac{2s}{(2-r^2)} \cdot \frac{1}{\Gamma()}\int_0^r\int_0^t (t-z)\,\alpha - 1\ g(z, u(z))dz.dt,\ s \in I$$

We see that u ϵ C[0, 1] is a solution of (2.18) if and only if u ϵ C[0, 1] is the fixed point of the mapping T. Suppose the following conditions:

(J_1) $\exists\zeta: R^2 \rightarrow R$ and $\varphi \in \phi$ such that, for all s ϵ I and a, b ϵ R with ζ (a, b) ≥ 0,

$|g(s, a) - g(s, b)| \leq K_1\, \phi(|a - b|)$, $K_1 = \frac{\Gamma(+2)}{(5+3)}$;

(J2) ∃u_0 ϵ C[0, 1] such that ζ (u_0(s), T(u_0(s))) $\geq$ 0 for all s ϵ I;

(J3) $\forall$ s ϵ I and u, v ϵ C[0, 1],

ζ (u(s), v(s)) $\geq$ 0 implies that ζ (T(u(s)), T(v(s))) $\geq$ 0

(J4) Let (u_n) be a sequence in C[0, 1] such that $u_n \to$ u in C[0, 1]. Let, for all t ϵ I,

ζ (u_n(s), u_{n+1}(s)) $\geq$ 0 ∀n ϵ N $\Rightarrow$ ξ (u_n(s), u(s)) $\geq$ 0.

Theorem 3.1 Let conditions (J1) -(J4) be fulfilled. Then T has at least one fixed point u* ϵ C [0,1].

Proof: We prove that T is a generalized θ-p-ϖ contraction mapping. Now, let u, v ϵ C[0, 1] such that, for all s ϵ I, ζ (u(s), v(s)) $\geq$ 0. Applying (J1),

$$|T(u)(s) - T(v)(s)| = \frac{1}{\Gamma()}\int_0^t (s-t)\,\alpha - 1\text{g(t, u(t)) dt} - \frac{2s}{(2-r^2)}\cdot\frac{1}{\Gamma()}\int_0^1 (1-t)\,\alpha - 1\text{g(t, u(t)) dt}$$

$$+\frac{2s}{(2-r^2)}\cdot\frac{1}{\Gamma()}\int_0^r\int_0^t (t-z)\,\alpha - 1\text{g(z, u(z))dz.dt} - \frac{1}{\Gamma()}\int_0^t (s-t)\,\alpha - 1\text{g(t, v(t)) dt}$$

$$+\frac{2s}{(2-r^2)}\cdot\frac{1}{\Gamma()}\int_0^1 (1-t)\,\alpha - 1\text{g(t, v(t)) dt} + \frac{2s}{(2-r^2)}\cdot\frac{1}{\Gamma()}\int_0^r\int_0^t (t-z)\,\alpha - 1\text{g(z, v(z))dz.dt}$$

$$\leq \frac{1}{\Gamma()}\int_0^t |s-t|\,\alpha - 1|g(t,u(t)) - g(t,v(t))|dt$$

$$+\frac{2s}{(2-r^2)}\frac{1}{\Gamma()}\int_0^1 |1-t|\,\alpha - 1|g(t,u(t)) - g(t,v(t))|dt$$

$$+\frac{2s}{(2-r^2)}\frac{1}{\Gamma()}\int_0^r \left|\int_0^t ((t-z)^{\alpha-1})\right| g(z,u(z)) - g(z,v(z)|dz|dt$$

$$\leq \frac{1}{\Gamma()}\int_0^t |s-t|\,\alpha - 1K_1\phi(|v(t)-u(t)|)dt$$

$$+ \frac{2s}{(2-r^2)}\ \frac{1}{\Gamma()}\ \int_0^1 |1-t|^{\alpha-1}K_1\phi\,(|v(t)-u(z)|)dt \ + \ \frac{2t}{(2-r^2)}\cdot\ \frac{1}{\Gamma()}\int_0^r (\int_0^t |t-z|^{\alpha-1}K_1\phi(|v(z) - u(z)|)dz)dt$$

$$\leq K_1\phi(\|v-u\|_\infty) \times \sup_{s\in(0,1)}(\frac{1}{\Gamma()}\int_0^t |s-t|^{\alpha-1}\,dt$$

$$+\frac{2s}{(2-r^2)}\frac{1}{\Gamma()}\int_0^1 |1-t|^{\alpha-1}\,dt + \frac{2s}{(2-r^2)}\frac{1}{\Gamma()}\int_0^r\int_0^t |t-z|^{\alpha-1}\,dzdt$$

$$\leq \phi(\|v-u\|_\infty) = \phi(d(u,v))$$

We define β : C[0,1] x C[0,1]→ [0,∞) by

$$\beta\,(\text{u,v}) = \begin{cases} 1, & (u(s),v(s)) \geq 0\ \forall\ s \in I, \\ & 0,\text{otherwise,} \end{cases}$$

and

$$\text{p(u,v)} = \begin{cases} \frac{\phi(d(u,v))}{d(u,v)} & \text{if u} \neq \text{v}, \\ 0, \text{if u} = \text{v}, \end{cases}$$

Then, for all u, v ϵ C[0,1], we have

$$\beta(u,v)d(Tu,Tv) \leq \phi(d(u,v)) = \frac{\phi(d(u,v)}{d(u,v)}d(u,v)$$
$$= p(u,v)d(u,v)$$
$$\leq p(u,v)R(u,v).$$

Let ϖ (x)=x for all x ϵ [0, ∞). Then T is a generalized θ-p- ϖ reduction type mapping. Ace can show that all the suggestions of Theorem 2.6 are satisfied. Thus there exists u^* ϵ C[0,1] such that $Tu^* = u^*$.

APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

Let U = C[0,1] be the space of all constant functions defined on I, Where I = [0,1] and u ϵ U. Study the following two-point boundary value problem of a second order ordinary differential equations.

$$\begin{cases} -u^n(s) - g(s,u(s)) = 0; \\ \quad u(0) = u(1) = 0, \end{cases} \qquad s \in [0,1], \tag{4.1}$$

wherever$g: [0,1] * R \to R$is a continuous function. It is acknowledged that u=u(s) ϵ C[0,1] is a solution of 4.1 if and only if u ϵ C[0,1] is a answer of the integral equations.

$$u(s) = \int_0^1 k(s,t)g(t,u(t))dt,$$

Where $k(s,t)$ is defined as follows:

$$k(s,t)\begin{cases} s(1-t) & 0 \leq s \leq t \leq 1, \\ (1-s) & 0 \leq t \leq s \leq 1, \end{cases}$$

Theorem 4.1 Let $\varphi \in \phi$ and the following conditions be satisfied:

(J_1) ∃a function $\zeta: R^2 \to R$such that, for all $s \in I$and for all $a, b \in R$ with $\zeta(a,b) \geq 0$, we have $|g(s,a) - g(s,b)| \leq 8\phi(|a-b|)$;

(J_2) $\exists\, u_1 \in C[0,1]$ such that, for all$s \in I$,

$$\zeta(u_1(s), \int_0^1 k(s,t)g(t,u_1(t))dt) \geq 0$$

(J_3) For all $s \in I$ and u, v ϵ C[0, 1]

$\zeta(u_1(s), v(s)) \geq 0$ implies

$$\zeta(\int_0^1 k(s,t)g(t,u(t))dt, \int_0^1 k(s,t)g(t,v(t))dt,) \geq 0;$$

(J_4) Let {u_n} be a sequence in C[0,1] such that $u_n \to u$in C[0,1]. Let, for all $s \in I$ and $n \in N$, $\phi(u_n(s), u_{n+1}(s)) \geq 0$ imply that $\zeta(u_n(s), u(s)) \geq 0$.

Then boundary value problem (4.1) has a solution.

Proof: We define $T: C[0,1] \to C[0,1]$ by

$$T(u(s)) = \int_0^1 k(s,t)f(t,u(t))dt, \forall s \in I.$$

A solution of problem (4.1) look like to fixed point of T. Now our purpose is to verify that integral operator T is a generalized α-p- ϖ contraction.

Let u, v ϵ C[0,1] such that $\zeta(u(s),v(s)) \geq 0$ for all $s \in I$. Applying (J_1),

$$|Tu(s) - Tv(s)| = \left|\int_0^1 k(s,t)(f(t,u(t)) - f(t,y(t)))dt\right|$$

$$\leq \int_0^1 k(s,t)|g(t,u(t)) - g(t,y(t))|dt$$

$$\leq \int_0^1 k(s,t)(8\phi(|u(t) - v(t)|)dt$$

$$\leq 8\sup_{s\in I} \int_0^1 k(s,t)dt\phi(d(u,v))$$

$$= \phi(d(u,v))$$

We define $\beta: C[0,1] * C[0,1] \to [0,\infty)$ by

$$\beta(u,v) = \begin{cases} 1 & if \zeta\big(u(s),v(s)\big) \geq 0 \\ 0 & otherwise \end{cases} for\ all\ s \in I,$$

And

$$p(u,v) = \begin{cases} \frac{\phi(d(u,v))}{d(u,v)} & if\ u \neq v, \\ 0 & if\ u = v, \end{cases}$$

Then, for all u, v ϵ C[0,1], we have

$$\beta(u,v)d(Tu,Tv) \leq \phi(d(u,v)) = \frac{\phi(d(u,v))}{d(u,v)}d(u,v)$$

$$= p(u,v)d(u,v)$$

$$\leq p(u,v)R(u,v).$$

Let ϖ *(x)*=x for all $x \in [0,\infty)$. Then T is a generalized α-p- ϖ contraction type mapping.

Let $\{u_n\},\{v_n\}$ be sequences in C[0,1] such that $\lim_{n\to\infty} p(u_n,v_n) = 1$ and, for all $n \in N$, $\alpha(u_n,v_n) \neq 0$. By the definition of β, for all $n \in N$, and $t \in [0,1]$, we have $\zeta(u_n(s),v_n(s)) \geq 0$, and so $d(Tu_n(s),Tv_n(s)) \leq \varphi(d(u_n,v_n))$, which implies that, for all $n \in N$, $d(Tu_n,Tv_n) \leq \phi(d(u_n,v_n))$. Since $\lim_{n\to\infty} d(u_n,v_n) = 0$, therefore $\lim_{n\to\infty} d(Tu_n,Tv_n) = 0$. This suggests that condition (i) of Theorem 2.6 is fulfilled. Put on conditions (J_2)-(J_4), all the hypotheses of Theorem 2.6 are fulfilled. Therefore, there exists $u^* \in C[0,1]$ much that $Tu^* = u^*$.

CONCLUSIONS

Our contribution involves the enlargement and unification of findings from the present body of literature. We have utilized our fixed point findings in establishing solutions for ordinary Differential equation and fractional operators containing singular kernels, specifically the Caputo-type fractional operator. We included an integral-type boundary condition for the Caputo fractional boundary value problem, but not for the Caputo-type fractional boundary value problem. Our future investigate will spread these methods to fractional operatives with nonsingular kernels.

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