

# Differential Equations in Quantum Mechanics and Schrödinger Equation

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## Abstract

*In this study we investigate how differential equations play a role in quantum mechanics, focusing mainly on the role of a prominent one, Schrödinger's equation, which governs the behavior of quantum systems. In time two the mathematical picture of the wave nature of particles at quantum scales is provided, and the resulting equations are Schrödinger's equation, both in its time dependent as well as in its time independent form. The time dependent Schrödinger equation is used to describe how a quantum state evolves over time, whereas the time independent version is used to analyze systems that have fixed energy levels like particles in a potential well. Crossing solutions to Schrödinger's equation produce these wave functions, a treasure trove of information as to how a particle will behave, and even allow us to calculate measurable such as energy and angular momentum. In addition, the study discusses how to apply boundary conditions and potential functions to solve Schrödinger's equation to specific quantum systems such as the harmonic oscillator and hydrogen atom. When we solve these differential equations, we get deeper insight into fundamental quantum phenomena, which is useful for future progress in quantum computing, nanotechnology, and atomic physics.*

**Keywords:** Quantum mechanics, schrödinger's equation, differential equations, time-independent

## INTRODUCTION

In the early 20th century, quantum mechanics took us a step further, further driving our understanding of matter and energy at atomic, and subatomic scales, providing a framework for understanding things that classical mechanics failed to explain. Discoveries that contradicted classical physics laid historical foundations of quantum mechanics; such quanta began with Max Planck's solution to blackbody radiation in which he introduced quantized energy levels. It almost set things straight, though soon Albert Einstein's explanation of the photoelectric effect reinforced that light could be particle-like, and heralded a change in the public's understanding. More momentum to this transition was given with Niels Bohr's model of the hydrogen atom, proposing discrete, electron orbits and later, de Broglie's hypothesis that particles could be wave like, creating the wave particle duality. Among key principles that form the basis of quantum mechanics, superposition principle states that particles are simultaneously present in several states, and uncertainty principle based on ideas of Werner Heisenberg, when a particle is, for example, tightly localized in space, its momentum is correspondingly less tightly defined and vice versa. Quantum mechanics also asserts probabilistic rather than deterministic outcomes and necessitates rethinking of measurement and of what is real.

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The Schrödinger equation, a fundamental differential equation at the heart of quantum mechanics, describes how the quantum state of a system changes over time. The dynamic systems like these give rise to mathematical descriptions via differential equations which are required in quantum theory. For an instance, the Schrödinger

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equation can be a partial differential equation that will give a wave function of a particle (giving us the energy of the particle as well as the spatial distribution). Solving the Schrödinger equation for many different potentials allows physicists to predict electron configurations, molecular bonds, and behavior in quantum fields. Concepts like wave functions, energy eigenvalues, the probabilistic distribution of particles, and the other quantum mechanical models that cross the landscape as we consider atomic physics, chemistry, and quantum computing are all supported by differential equations. For this reason differential equations are indispensable tools for allowing scientists to bridge theoretical quantum principles and observed phenomena in order to drive the development of technologies based on quantum mechanics.

### Mathematical Foundations

*Complex numbers, linear:* The reason that complex numbers are fundamental is that they consist in the representation of wave functions in which the probabilities amplitudes often have imaginary components, which are necessary to describe oscillatory and interference phenomena in the behaviour of quantum states. Like it, linear algebra is equally crucial; it provides us with the language of vectors and matrices, which we use to do math with quantum states and observables. Hilbert spaces play an important role in quantum mechanics as they provide a mathematical structure for system states, represented as vectors, within a space equipped with an inner product structure that can produce distances and angles between those system states, necessary to calculate probabilities and orthogonally. Infinite dimensions require a robust framework to generalize finite dimensional concepts, and so Hilbert spaces provide this framework, in particular because they can accommodate the continuous variables of the position and momentum of a quantum particle. They have a crucial role here, as operators, that is, transforms on these quantum states. Operators which act on wave functions to give position and momentum are functions of position and momentum and act in a Hilbert space. Each observable in quantum mechanics is matched to a Hermitian operator and the eigenvalues of those operators are quantized, measurable quantities like energy or angle momentum. Its eigenvalues and eigenvectors reveal what the possible measurements the operators can make, and what probability distributions are associated with these measurements. Mathematical underpinning of quantum mechanics relies on combining complex numbers, linear algebra and Hilbert spaces in order to describe and predict the behavior of quantum systems well.

### Differential Equations in Physics

When you wish to model dynamic systems and run some changes in space and time, differential equations are crucial tools in physics. They arise naturally in physical models by the way that quantities evolve under forces, fields or potentials. Differential equations come in two primary forms: Our focus is on ordinary differential equations (ODEs) and partial differential equations (PDEs). Odes are functions on 1 variable and their derivatives, and often are simpler systems, e.g. motion of particle in 1 dimension, under Newton's laws. PDEs are needed to model more complicated systems with more space and time variables than in the case of PDEs, e.g., fluid flow, heat conductivity, etc. by comparison. A PDE in physics is one that has quite a unique example as a foundational PDE, i.e., Schrödinger's equation, which expresses how quantum states evolve. Boundary and initial conditions are sought in order of differential equations to generate interesting physical solutions. By specifying the behavior of a system on the edges of a domain, such as the ends of a vibrating string fixed at certain points, boundary conditions define the system behavior; initial conditions describe the state of a system at a starting time: for instance, the initial velocity and position for a projectile. Together these conditions ensure unique and physically relevant solutions for real world systems and provide additional clarity in the modeling of such systems. They disentangle, once translated by physicists of physical laws into definite differential equations with well-defined boundary and initial conditions, into powerful clever predictive tools available for inestimable understanding of many different phenomena: from planetary motion to electromagnetic waves or to the behavior of atoms.

### Introduction to Schrödinger's Equation

In 1925, Austrian physicist Erwin Schrödinger made his Schrödinger equation, one of the corner

stones of quantum mechanics. Doing so gave a mathematically bold, pioneering way to describe particles as behaving at the scale of the atomic and subatomic scales, different from classical physics by its wave functions. Schrödinger's equation describes the probabilistic behavior of quantum systems, in which electrons do not, in fact, follow exact paths, but exist in probabilistic clouds.

$$i\hbar \frac{\partial \psi(r,t)}{\partial t} = \hat{H}\psi(r,t)$$

$i$  is the imaginary unit,

$\hbar$  is the reduced Planck's constant,

$\psi(r,t)$  is the wave function, depending on position  $r$  and time  $t$ .

$\hat{H}$  The Hamiltonian operator, which represents the total energy (kinetic and potential) of the system. This equation is fundamentally important, it describes how quantum states evolve over time, which is the quantum analogue to Newton's laws in classical mechanics. The Schrödinger equation exists in two main forms: time dependent and time independent equations. If one has some system that doesn't want to sit around for long, the time dependent Schrodinger equation describes that motion of the quantum state, the quantum state of the quantum system as it changes in time. The stationary state is itself represented by the time independent Schrödinger equation, and this is due to the systems we consider will have fixed energy and hence the stationary state is stationary. This form is essential to understanding atomic structure and molecular bonding and essential to energy level calculations in quantum systems. The Schrödinger equation has a physical interpretation that revolves around  $\Psi$  (the wave function), the set of information about its quantum system's state. It is probabilistic rather than deterministic nature of quantum mechanics, the probability density of finding a particle in a particular location is represented by the square of the wave function's magnitude ( $|\Psi|^2$ ). The Schrödinger equation can be solved for a particular system and insight is gained into the possible configurations and energy states that the system can adopt. For example, solutions of the Schrödinger equation show quantized energy levels for electrons in atoms, and are used to predict chemical and physical properties. Working in summary, quantum mechanics is based upon the Schrödinger equation, which describes the evolution of quantum states and gives deep understanding of the nature of matter and energy at its smallest scales.

### Time-Independent Schrödinger Equation

The time independent Schrodinger equation is a very fundamental equation in quantum mechanics where the does not change with time implying the energy is constant.

$$\hat{H}\Psi(r) = E\Psi(r)$$

$\hat{H}$  is the Hamiltonian operator, which represents the total energy (kinetic plus potential) of the system,  
 $\Psi(r)$  is the wave function, depending on position  $r$ , and describes the probability distribution of the particle's position,

$E$  is the total energy of the system, which is a constant. Hamiltonian operator and  $\Psi$  is the wave function. So typically the Hamiltonian operator  $H$  is represented with  $H$  just kinetic and potential energy terms and the time independent Schrodinger equation then describes what energy levels of particles in that potential field are in. Values for this equation offer important information into the quantum behavior of particles trapped in a variety of potential wells and barriers. The purpose of the equation is that for example, in the case of a potential well, where the particles are confined to a limited region, the equation provides quantized energy levels, instead of continuum, the particles are allowed to occupy certain discrete energy states. The time independent Schrödinger equation, when used on potential barriers, explains things like quantum tunneling where particles, although classically impossible, have a nonzero probability of passing barriers. Among the key applications of the time independent Schrödinger equation we mention the modeling of the "particle in a box" and the "quantum harmonic oscillator". In the particle in a box model, a particle is constrained to a one dimensional, rigid box with infinite walls in which it is permitted to only a certain wavelengths — and therefore only specific energies — available in the box. This is a simple, yet powerful model for understanding electrons in

atoms and molecules. Another classic application is of the quantum harmonic oscillator to model particles under a parabolic potential, such as atoms vibrating in a molecule. Focusing on the molecular vibration, the particle's vibrational modes correspond to equidistant quantized energy levels that are crucial to describing molecular vibrations and phonons in solid state physics. Therefore, the time independent Schrödinger equation is important to gain insight into quantum system, in providing the values of quantized states which govern the structure of atoms, the formation of chemical bonds, and many other primitive quantum mechanics phenomena.

### Time-Dependent Schrödinger Equation

A fundamental equation in quantum mechanics, the time dependent version of Schrödinger equation gives the time dependent behaviour of the quantum state of the system. In this case it is reductive from the way how the wave function  $\Psi(x, t)$ , which represents the probability amplitude for a particle's position and time, is necessarily subjected to a dynamic equation containing spatial and temporal variations. Mathematically, it is expressed as  $i\hbar \frac{\partial \Psi(x,t)}{\partial t} = H\Psi(x, t)$ , where  $i$  is the imaginary unit,  $\hbar$  is the reduced Planck's constant,  $H$  is the Hamiltonian operator (representing total energy), and  $\Psi(x, t)$  is the time-dependent wave function. This form is applicable to any quantum system subject to time evolution, as it dictates how a particle's wave function changes under the influence of various forces or potentials. The wave function  $\Psi(x, t)$  plays a fundamental role in interpreting quantum mechanics, as its square magnitude,  $|\Psi(x, t)|^2$  represents the probability density of finding a particle at position  $x$  and time  $t$ . The classical interpretation where objects have definite trajectories is thus contradicted by this probabilistic interpretation. In quantum mechanics, however, the wave function is only a description of the probability of locating a particle at a particular position and time making uncertainty and superposition the beating heart of quantum behavior. Time dependent Schrödinger equation is essential to understand how particles in motion such as free particle and quantum tunneling. When considering a free particle, for which no potential field confines the movement of the particle, solutions of the time dependent Schrödinger equation give rise to traveling wave functions describing the particle dispersion with time. Quantum wave packet evolution is demonstrated in this scenario and allows some insights into particle wave duality.

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{x})\psi = i\hbar \frac{\partial \psi}{\partial t}$$

Dividing by  $\psi$ , the equation separates as

$$-\frac{\hbar^2}{2m} \frac{\nabla^2 \psi}{\psi} + V(\mathbf{x}) = i\hbar \frac{f'}{f}$$

The left-hand side of the equation depends only on  $x$ , while the right-hand side depends only on  $t$ . Therefore, both sides must be independent of  $x$  and  $t$  and must be equal to a constant. We designate this separation constant as  $E$ , thereby reintroducing the concept of total energy into the equation. This results in two separate differential equations, each depending on only one variable.

$$f' = -\frac{iE}{\hbar} f, \quad -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{x})\psi = E\psi.$$

The second equation is known as the time-independent Schrödinger equation. The first equation can be readily integrated to find a solution.

$$f(t) = e^{-iEt/\hbar},$$

Which can be multiplied by an arbitrary constant.

A quantum effect that is fascinating in its own way is quantum tunneling in which a particle will have a probability of passing through a potential barrier when its energy isn't sufficient for it to overcome classically. The explanation for this phenomenon based on the time dependent Schrödinger equation has, among other uses, important application in electron tunneling in semiconductors and nuclear fusion

in stars. As found through these applications, quantum mechanics does not behave in a way that is intuitive, and the time dependent Schrödinger equation serves to illustrate such non intuitive property of a quantum particles in that they can act outside of the classically expected behavior for example, in this manner, particles will behave as waves with probabilities.

### **Analytical and Numerical Solution Techniques**

The solution of the Schrödinger equation for complex quantum systems often requires a combination of analytical and numerical approaches with the first providing qualitative insights and the second quantitative approximations. Another primary analytical technique is separation of variables, where the time dependent Schrödinger equation is simplified in the assumption that the wave function  $\Psi(x, t)$  can be written as a product of position and time dependent functions, e.g.  $\Psi(x, t) = \psi(x)\phi(t)$  and provided the boundary conditions are fixed, and the potentials are symmetric this approach is particularly successful, as it is able to convert the time dependent Schrödinger equation into an exact time independent one, The second widely used analytical technique is perturbation theory, which is used when a total solution to the Schrödinger equation is not practical because of small perturbations of the potential. Perturbation theory treats these perturbations as adjuncts to a known solution system by finding an approximate solution that reveals how changes in energy levels and states of a system are affected by a small change in its potential. In atomic and molecular physics such behavior is not too far off the mark and the success of this approach is well documented. Numerical approaches are needed for accurate solution of many complex quantum problems. Discretizing the continuous Schrödinger equation into a series of difference equations on a grid, the finite difference method is a key numerical technique. The numerical solution to the wave function over discrete intervals is done by this approach, which approximates derivatives as a difference between points on a grid. Because analytical solutions are infeasible for systems with complicated potentials such as multi-electron atoms and solid state materials, finite difference methods are especially useful. The curve fitting and Schrödinger's equation are two independent methods for solving the problem, each with a few other numerical methods available, including the finite element method and the Runge-Kutta method, that expand the toolkit for solving Schrödinger's equation on diverse quantum systems and boundary conditions. By working together, these analytical and numerical approaches empower physicists to address a broad family of quantum problems, from simple particle in a box systems to intricate, insolvable systems, revealing new quantum behavior and its applications in technology and materials science.

### **Quantum Field Theory Overview**

As a result, the remaining limitations were addressed and interactions were described in a framework including both particle creation and annihilation, by extending quantum mechanics to quantum field theory (QFT). In QFT, particles are the excitations / quanta of underlying fields and each fundamental particle has its own corresponding quantum field. The Schrödinger and Dirac equations treat the particles as having fixed identities; QFT redefines the particles themselves as transient manifestations of fields and it does in a way that reproduce the observed phenomena of high energy physics.

The starting point in QFT is typically the Lagrangian density,  $L$ , which encapsulates the dynamics of the fields. It's an extension of the Lagrangian in classical mechanics, designed to account for both spatial and temporal variations of fields. For a scalar field  $\phi(x)$ , the Lagrangian density is given by:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi)$$

Where  $\partial_\mu$  represents the derivative with respect to spacetime coordinates,  $m$  is the particle's mass, and  $V(\phi)$  is the potential energy.

The first application of quantum mechanics combined with one of the four fundamental forces was quantum electrodynamics (QED)—not a fully fledged quantum field theory (QFT), but a subset of QFT that was able to successfully describe how charged particles and photons interact (with an accuracy unprecedented by all other physical theories). In QFT, forces are mediated by field quanta: In the case

of electromagnetism it's just photons, for the strong force it's gluons, for the weak force it's W/Z bosons. This framework is also the basis of the Standard model of particle physics: the description of the interactions of all elementary particles. Processes like electron positron pair production, particle scattering and the behavior of the quarks and gluons inside a proton or a neutron are explained in QFT. While Schrödinger's equation is a powerful model for non-relativistic quantum systems, its range is restricted and its model is incomplete, and extensions are needed to deal with the many quantum phenomena occurring in nature. Relativistic quantum mechanics is started with the Dirac equation which introduces relativistic corrections and predicts antimatter. Quantum field theory extends our understanding by viewing particle as field excitations that may create and annihilation, and that describe fundamental interactions with high precision. These advances together expand the range of applicability of quantum mechanics, which can now predict complex, high energy systems with better accuracy, and provide a window on the universe's most basic processes.

### Heuristic Derivation of the Schrödinger Equation

Heuristically combining classical wave mechanics and underlying laws of quantum physics one can derive the Schrödinger equation, a foundational equation in quantum mechanics. A rigorous derivation usually necessitates the use of advanced postulates of quantum mechanics, yet a heuristic treatment provides an intuitive view of the reasoning behind its formulation. In this work we connect de Broglie's hypothesis of wave particle duality, the energy momentum relationship, and wave dynamics.

### Wave-Particle Duality and de Broglie's Hypothesis

The concept of wave-particle duality, introduced by Louis de Broglie, posits that all matter has both wave and particle characteristics. For a particle with momentum  $p$  and energy  $E$ , de Broglie associated a wavelength  $\lambda$  and frequency  $f$  through the relations  $\lambda = h/p$  and  $f = E/h$ , where  $h$  is Planck's constant. In quantum mechanics we have a wave function  $\Psi(x, t)$  (a function of a complex number that contains all the information of the system). First, we relate the wave properties of this wave function to classical motion and momentum and energy for the particle.

### The Classical Wave Equation and Quantum Adaptation

For a wave propagating in space, the classical wave equation takes the form  $\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$ , where  $v$  is the wave speed. In quantum mechanics, however, the relationship between frequency and wavelength needs adjustment to account for the probabilistic and discrete nature of particle states. By introducing the de Broglie relations and the wave's wavenumber  $k = 2\pi/\lambda$  and angular frequency  $\omega = 2\pi f$ , we can express the wave function for a free particle as  $\Psi(x, t) = Ae^{i(kx - \omega t)}$ , where  $i$  is the imaginary unit, and  $A$  is an amplitude constant. Substituting de Broglie's expressions, we have  $k = p/\hbar$  and  $\omega = E/\hbar$ , where  $\hbar = h/2\pi$  is the reduced Planck constant. Consequently, the wave function for a free particle becomes  $\Psi(x, t) = Ae^{i(px/\hbar - Et/\hbar)}$

### Energy and Momentum Operators

A key insight in quantum mechanics is that energy and momentum are represented as operators acting on the wave function. The momentum operator, derived from the relation  $p = \hbar k$ , is given by  $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ , while the energy operator, from  $E = \hbar \omega$ , is  $\hat{E} = i\hbar \frac{\partial}{\partial t}$ . Applying these operators to the wave function yields the values of momentum and energy, capturing the particle's dynamical properties in a way compatible with wave mechanics.

### Heuristic Construction of Schrödinger's Equation

To derive Schrödinger's equation, we consider a particle moving in a potential  $V(x)$ , with total energy  $E = p^2/2m + V(x)$  for non-relativistic particles. Replacing  $E$  and  $p$  with their operator forms, we get the operator equation:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x,t),$$

Where  $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$  represents the kinetic energy term, and  $V(x)\Psi(x,t)$  represents the potential energy acting on the particle. This equation, known as the time-dependent Schrödinger equation, dictates how the wave function  $\Psi(x,t)$  evolves over time in the presence of a potential.

### Interpretation and Implications

Quantum mechanics' probabilistic nature is encapsulated in the time dependent Schrödinger equation. This equation governed by the wave function  $\Psi(x,t)$  rather than certainties about a particle's position and momentum gives probabilities only rather than certainties. The probability density of finding the particle at position  $x$  and at time  $t$  is represented by the square of the modulus of that wave function,  $|\Psi(x,t)|^2$ . There it bridges the divide between the deterministic paths a particle should take with classical mechanics, and the fundamental probabilistic behavior of particles in quantum mechanics. In short, this heuristic derivation brings in the ideas of wave particle duality, energy momentum relations and wave dynamics together to lead to the Schrödinger equation. Schrödinger's equation integrates wave characteristics and operators for momentum and energy, and gives predictions of how quantum systems will behave. Being quantum mechanical, you can become any path; however the equation in which you exist is not only at the center of atomic and molecular physics but also has been a foundation of fields such as quantum computing and cryptography and quantum field theory, showing the generality and profound consequences of quantum mechanics.

### LITERATURE REVIEW

*Barley, K et al. (2022).* Schrodinger's equation is a fundamental equation of quantum mechanics which describes how the quantum state of a physical system varies in time. Formulated by physicist Erwin Schrödinger in 1926, it is a mathematical foundation for how particles, like electrons in atoms, respond at the quantum level. It's really kind of saying that particles are waves and we can think of them in terms of probability of being in this or that state. For a single particle, it is a partial differential equation, containing terms for the particle's kinetic and potential energy. The solution of the equation, called the wave function, tells us of the probability of a particle's location and momentum, indicating its probabilistic character. In quantum mechanics, the Schrödinger Equation is critical, for predicting atomic and molecular behavior and for developing much of modern physics and chemistry.

*Reed, B. C. (2022).* Classical physics hits the wall as soon as you begin the branch of physics that explores the behavior of matter and energy at atomic and subatomic levels; it is called quantum mechanics. The Schrödinger Equation and the path integral formulation are two central concepts of quantum mechanics. In this post, I will discuss the Schrödinger Equation developed by Erwin Schrödinger, which is a formula for calculating the evolution of a given systems wave function (the probabilities of a partical being found somewhere, having some momentum, etc.) Clear understanding of structure of an atom, molecule and small part of a solid is based on it. An alternative approach is provided by Richard Feynman's path integral formulation of particles who do not have single trajectories, but instead take all possible paths between initial and final points. The correlated Hermitian part of this method computes probabilities as sums of contributions from all possible paths, illustrating the uncertainty inherent to quantum behavior. By taking these concepts together, we complement with complementary perspectives of particles being probabilistic and wave like in quantum systems.

*Mita, K. (2021).* Schrödinger's is similar structurally to, but it's about something slightly different: The diffusion equation. Formulated in Schrödinger's equation, it is used to calculate the evolution of the wave function that describes a quantum system's probability distribution for the position and momentum of the particle. As it turned out, Schrödinger's equation itself looks like a diffusion equation, which is the equation that describes how, say, particles or heat spread in time. Unlike the real valued diffusion equation, which describes spreading of concentration from high to low concentrations

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gradually, Schrödinger's equation has complex numbers that allow oscillatory solutions -- like waves rather than waves of particles.

*Nagasawa, M. (2012).* Systems changing over time are described by the Schrödinger Equation and by diffusion theory, but in different realms. In quantum mechanics, the Schrödinger Equation is fundamental, described it is how a system's wave function changes: probabilities to be in one or more states, a particle's position or momentum for example. The wave-like nature of particles implies that they are probabilistic, not deterministic, and so the wave-like nature of particles is captured. Diffusion theory, used for instance in biology and physics, namely the spread of particles, heat or substances through a medium at macroscopic scale, contrasts with the approach done in this dissertation. Like the form of the Schrödinger Equation, diffusion equation describes probabilistic flow or dispersal of particles from regions of high to low.

*Mohebbi, A et al (2013).* In quantum mechanics the equivalence postulate states that all physical states of a quantum system can be described in a single unified framework. It proposes a boost of the traditional quantum formalism, where it attempts (without making any specific assumption of a privileged reference frame or classical background) to define quantum mechanics. At a fundamental level, these potential representations or configurations of a system are equivalent, and this approach implies. The postulate gives rise to the idea of 'quantum states' which are invariant under transformation which in fact leads to the idea that all states of a system are related by symmetry transformation. The key quantum concept underlying the use of this principle and granting it the meaning it has is gauge invariance, and it is meant to explain how particles' behavior can be described consistently across different contexts.

*Mohebbi, A., et al (2012)* Quantum mechanics' equivalence postulate states, amongst other things, that each physical state of a quantum system can be described in terms of a single, unitary framework. This is proposed as an extension to the traditional quantum formalism in which the quantum formalism itself can be derived without the prior assumption of a privileged reference frame or a specific classical background. The fundamental undertaking of this approach is to infer that different potential representations or configuration of a system, for example, position or momentum, are all equivalent. This postulate is equivalent to that of quantum states that are themselves invariant under transformation, i.e. all states of a system can be shown to be equivalent by transformation. With respect to its interpretive emphasis on dynamics, this principle parallels important quantum concepts such as gauge invariance and thus provides a natural framework for making sense of characteristic particle behavior in disparate settings while maintaining consistency in the way in which they are described.

*Nagasawa, M. (2012).* The fractional Schrödinger equation is the generalization of the traditional Schrödinger equation including fractional calculus but allows to describe quantum phenomena with anomalous diffusion or nonlocal interactions. In this modification, a fractional Laplacian operator is introduced to replace the traditional second order spatial derivative with one involving a derivative of non-integer order. The equation models quantum particles that perform Lévy flight or exhibit slow spatial correlations, and therefore cannot be described by the equation typically used to model quantum systems. The fractional Schrödinger equation describes dynamics that cannot be accounted for by the usual Schrödinger framework, e.g. the case of fractal systems or complex media whose path of the particle departs from the conventional Brownian motion. The ability to generate tunnels of edge states, with the possibility of communicating quantum information through the medium of quantum excitations forming currents, provides a powerful means to explore new quantum behaviors, such as tunneling effects and wave propagation in disordered or heterogeneous environments, and deeper insights into the workings of underlying complex quantum systems.

### **Applications of Schrödinger's Equation and Differential Equations**

A diverse body of scientific and engineering applications depends on differential equations and

Schrödinger's equation. These are foundational tools for understanding and modelling complex systems across a wide spectrum of fields.

### ***Quantum Mechanics and Chemistry***

Quantum mechanics is defined around Schrödinger's equation, which describes the behaviour of particles at atomic and subatomic scales. Solving the time independent Schrödinger equation enables scientist to predict quantized energy of electrons in atoms, molecules and solids. This has practical applications in chemical bonding, reaction rates, and molecular structures. These solutions are used in computational chemistry to design drug, develop materials science, and develop catalysts.

### ***Solid-State Physics and Semiconductors***

Schrödinger's equation is used in solid state physics to study electron behavior in crystalline structures, such as semiconductors. This is how we can understand and design electronic devices that range from transistors to solar cells and LEDs based on knowledge of Electron energy bands and band gaps. Such applications are applied in the realm of electronics, and the modern technology is due to them.

### ***Quantum Computing***

Quantum computing uses quantum states to do complex calculations, and at the center of the work lies Schrödinger's equation. These systems allow the researchers to analyze and control quantum states and entanglements, and in turn harness them to applications such as cryptography, data security and solving problems that classical computers have no powers to solve.

### ***Optics and Laser Physics***

Light matter interactions, e.g., laser physics, are described by differential equation such as Schrödinger's equation. It is used to predict light amplification and coherence in laser systems and has application in telecommunications, medical imaging and laser surgery. These equations are also used quantum optics to describe how photons behave for advanced imaging and sensing technologies.

### ***Material Science and Nanotechnology***

Schrödinger's equation allows us to study nanomaterial's (electronics in nanoscale), predicting properties like conductivity, magnetism, and thermal resistance. The applications to practical materials development are in the use of new materials for batteries, sensors and nanoelectronics, key to miniaturize and high performing devices.

### ***Nuclear Physics and Energy***

Particle behaviour is modeled by differential equations that describe atomic nuclei reactions for physicists. This impact on nuclear energy production and medical imaging technologies (such as MRI and PET scans) as well as radiation therapies is direct.

### ***Environmental and Climate Modeling***

Differential equations are used to model physical phenomena in areas such as meteorology, oceanography where temperature distribution in the ocean and atmospheric changes are used to forecast the changes in temperature of the earth and solutions for the conservation of the environment.

At its core, Schrödinger's equation and differential equations can act as a force for science, technology and medicine to advance by studying and engineering modeling physical systems.

### **Limitations and Extensions of Schrödinger's Equation**

While Schrödinger's equation is a foundational tenet of quantum mechanics, it proves too restrictive in these high energy or relativistic, or in cases of complex quantum systems with particle creation and annihilation. As a result of this the necessary gaps between classical quantum mechanics and relativity and quantum field theory were bridged with extensions. This is a discussion of constraints of

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Schrödinger's equation and what led to the Dirac equation and quantum field theory which expand our definition of fundamental particles and forces.

### Limitations of Schrödinger's Equation

In fact, Schrödinger's equation works great for describing low energy non relativistic systems like electrons in atoms and molecules. Unfortunately, it relies on the approximation that particles move slower than the speed of light, which limits its use for systems where relativistic effects are small. The equation does not include relativistic time dilation and length contraction in high energy environments or in systems where particles travel at almost the speed of light. It also does not naturally account, on its own, for the creation and annihilation of particles that is so prevalent in high energy physics and field interactions. For example, particle accelerators, like particle collides and decays, cannot use Schrodinger's equation to observe or disappear due to missing field based descriptions. A shortcoming of this approach is that it cannot incorporate the intrinsic spin of particles, an important aspect of magnetic interactions, fermionic behavior and matter structure. As a consequence, electrons with spin and relativity are insufficiently described by Schroedinger's equation.

### CONCLUSION

Finally, in quantum mechanics differential equations are fundamental through Schrödinger's equation, which is the main pillar of quantum theory that describes the probabilistic behavior of particles at the atomic and subatomic scale. Both dynamic and stationary states of a quantum system can be understood from Schrödinger's equation, time dependent and time independent. These equations describe the possible solutions of wave functions, which tell us where and what other properties particles might have like positions, momenta, etc. and enable us to predict and then measure outcomes that otherwise would be inaccessible. Schrödinger's equation can be solved under all such situations: particles in a potential well, the hydrogen atom etc. and we can do it by applying boundary conditions and calculating potential functions. These solutions help us understand many fundamental aspects of quantum, from quantized energy levels to the wave particle duality of all matter. The fact that Schrödinger's equation accurately predicts experimental results naturally leads to the extension and further application of quantum mechanics. This foundational framework provides a foundation for technology driven fields like quantum computing, material science and nanotechnology. In general, differential equations, as exemplified by Schrödinger, bring mathematics to bear on the physical properties of quantum systems, and thus deepen our understanding of the universe itself at its most basic level, and open the door to the creation of revolutionary technological applications.

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